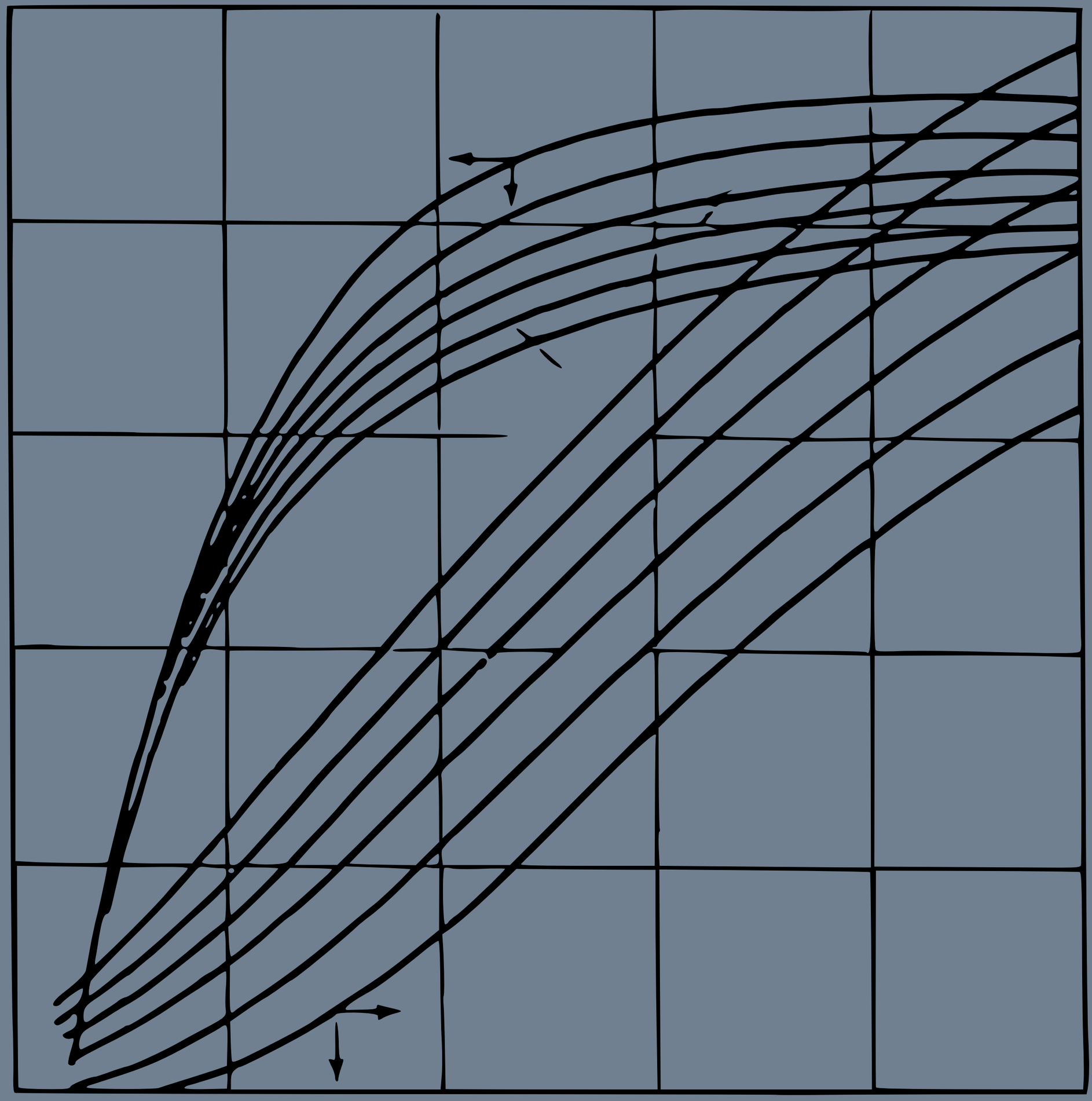
A.V. Luikov, Yu.A. Mikhailov

Theory of Energy and Mass Transfer



THEORY OF ENERGY AND MASS TRANSFER

BY

A. V. LUIKOV

AND

Yu. A. MIKHAILOV

Revised English Edition

TRANSLATED FROM THE RUSSIAN BY

L. A. FLNN

TRANSLATION EDITED BY

J. S. DUNN, M.A., PH. D., A. R. I. C.

AND

J. A. WEIGHTMAN, B.A. (HONS)

PERGAMON PRESS

OXFORD · LONDON · EDINBURGH · NEW YORK

PARIS · FRANKFURT

Pergamon Press Ltd., Headington Hill Hall, Oxford
4 & 5 Fitzroy Square, London W.1
Pergamon Press (Scotland) Ltd., 2 & 3 Teviot Place, Edinburgh 1
(Pergamon Press Inc., 122 East 55th St., New York 22, N.Y.
Gauthier-Villars, 55 Quai des Grands-Augustins, Paris 6
Pergamon Press GmbH, Kaiserstrasse 75, Frankfurt-am-Main

5
Copyright © 1965
PERGAMON PRESS LTD.

First English edition 196

Library of Congress Catalog Card Number 63—10012

This is an edited translation revised by the author, of the original Russian volume entitled *Теория переноса энергии и вещества (Teoriya perenosa energiyi i veshchestva)*, published in Minsk by the Publishing House of the Belorussian Academy of Sciences

Contents

PREFACE TO THE ENGLISH EDITION ix

FOREWORD 1

CHAPTER 1

The Fundamental Laws and Differential Equations of Molecular Transfer 5

1. Transfer Potentials 6
2. The Fundamental Relations of the Thermodynamics of Irreversible Processes 9
3. Onsager's System of Linear Equations 11
4. Differential Equations of Heat and Mass Transfer in Binary Gaseous Mixtures 14
5. Transfer Equations of Heat and Mass in Disperse Media in Case of Phase Change 22
6. The System of Differential Equations of the Transfer of Heat and Matter in Capillary-porous Bodies 27
7. Generalised System of Differential Equations 39

CHAPTER 2

Non-stationary Fields of Transfer Potential of Heat and Mass with Boundary Conditions of the Third Kind 49

1. Non-stationary Fields of Transfer Potential of Heat and Mass in Medium with Constant Temperature 52
2. Non-stationary Fields of Transfer Potential of Heat and Mass. Temperature of the Medium — a Linear Function of the Time 116
3. Non-stationary Fields of Transfer Potential of Heat and Mass. Temperature of Medium an Exponential Function of the Time 138

4. Generalized Solutions of the System of Equations for Heat and Mass Transfer 145
5. Special Cases of Heat and Mass Transfer. Certain Questions of Limiting Transitions in the System of Equations of Heat and Mass Transfer 152
6. Influence of the Criteria of Heat and Mass Exchange on the Process of Heat and Mass Transfer 171
7. Heat and Mass Transfer in the Presence of Phase and Chemical Transformation 186
8. Heat and Mass Exchange in a Layer of a Moist Body 193

CHAPTER 3

Non-stationary Fields of Transfer Potential of Heat and Mass with Boundary Conditions of the First Kind 197

1. Non-stationary Fields of Transfer Potential of Heat and Mass with Constant Mass Content of the Body 197
2. Heat and Mass Transfer with Invariable Mass Content of the Body and in the Absence of Phase Changes 215
3. Methods of Determining the Coefficients of Heat and Mass Transfer 228

CHAPTER 4

Non-stationary Fields of Transfer Potential of Heat and Mass under Boundary Conditions of the Second Kind 248

1. Infinite Plate ($\delta=0$) 249
2. Infinite Cylinder ($\delta=0$) 255
3. Sphere ($\delta=0$) 260
4. Analysis of the Solutions 265
5. Infinite Cylinder ($\delta\neq0$) 266
6. Sphere ($\delta\neq0$) 270
7. Infinite Plate. Generalized Boundary Conditions of the Second Kind 272
8. Certain Special Problems with Generalized Boundary Conditions 279

Chapter 5

Two-dimensional Fields of Heat and Mass Transfer Potentials 285

1. Finite Plate. Boundary Conditions of the Second Kind 285
2. Two-dimensional Plate. Generalised Boundary Conditions 290

Chapter 6

Heat and Mass Exchange in Molecular and Molar Transfer of Energy and Matter 313

1. Non-stationary Potential Fields of Transfer of Energy and Matter under Boundary Conditions of the Third Kind 314
2. Non-stationary Fields of Transfer of Energy and Matter under Boundary Conditions of the Second Kind 351
3. The Influence of the Criteria of Similarity on High Intensity Heat and Mass Exchange 357
4. Heat and Mass Transfer on Reduction of Pressure 365

Conclusion 378

List of Symbols 381

Bibliography 383

Preface to the English Edition

THE METHODS of the thermodynamics of irreversible processes have recently been more and more widely applied in the analytical study of the manifold phenomena of transfer. The system of differential equations developed for heat and mass transfer in disperse media in the presence of phase change has proved to be analogous to the system of differential equations of transfer in gaseous mixtures in the presence of chemical transformations. Hence the solutions obtained for the former system of equations of heat and mass transfer are of general interest for the study of the phenomena of molecular transfer. They can be regarded as an extension of the wel-known transfer theory of Onsager.

Since the publication of this monograph there have appeared in the Soviet Press a number of articles by the Authors which form a further development both of the general theory of transfer and of its individual sections. Certain material from these articles has already been published in monographs by one of the authors, but these monographs have appeared only in the Russian language. Hence for the sake of completeness of exposition and in order to take into account recent work on the theory of the transfer of energy and matter, the Authors have thought it essential to introduce certain additions and changes into the original text of the monograph.

The Authors express their thanks to the English Editors for so kindly agreeing to introduce this additional matter into the English translation of the book.

A. B. LUIKOV
YU. A. MIKHAILOV

Foreword

THE INVESTIGATION of many thermal and electrotechnical processes depends on the solution of problems of the molecular transfer of energy, mass and electricity. The exchanges of energy, heat and matter in solid bodies obey a simple linear law. For example, the transfer of heat by conduction is described by the linear law of Fourier: the specific heat flux is directly proportional to the temperature gradient. The transfer of electricity in metals obeys the linear law of Ohm: the specific flux of electricity is directly proportional to the gradient of the electric potential. The molecular transfer of matter is described by the law of diffusion: the specific flux of matter is directly proportional to the concentration gradient, etc.

These linear laws constitute the basis for the deduction of the corresponding differential equations (of heat conduction, electrical conduction and diffusion). The solution of these equations for definite initial and boundary conditions, characterizing the initial state of a body and the interaction of its surface with the surrounding medium, present difficulties which are well known. Hence, from the end of the eighteenth century right up to the present time, various methods have been worked out for the solution of the differential equations of heat conduction, electrical conduction and diffusion.

As a result of the development of experimental methods it has been shown that the phenomena of transfer are interconnected. For example, isothermal concentration diffusion in a gaseous mixture produces a transfer of heat and the formation of a temperature gradient (Dufour effect). Conversely, the transfer of heat by conduction in a gaseous mixture

produces a transfer of matter (thermo-diffusion, or Soret effect). A similar connection exists between the transfer of heat and the transfer of electricity. Hence the initial conditions on the basis of which the corresponding differential equations were deduced must nowadays be reconsidered by the methods of the thermodynamics of irreversible processes.

Instead of the separate, mutually independent linear equations of heat transfer, transfer of electricity and transfer of mass, we now assume, as the initial equations of transfer, the system of linear equations of Onsager, in which any particular form of transfer is characterized by the direct effect together with superimposed transfer phenomena. These linear equations of Onsager lead to a system of interconnected differential equations for the molecular transfer of heat, electricity and matter. Thus there arises the very important problem of working out methods for the solution of the system of differential equations for molecular transfer, which is more complicated than the problem of constructing the mathematical theory of heat conduction or diffusion.

The present monograph is devoted to working out the analytical theory of the transfer of heat, electricity and mass, i.e. to the creation of one of the most important branches of the analytical theory of the thermodynamics of irreversible processes.

So far as is known, no analogous research is to be found in the literature. The present monograph is written on the basis of the authors' own work alone.

The first part of the book is devoted to an exposition of the thermodynamics of the phenomena of transfer, and the deduction of a system of differential equations representing heat and mass transfer in capillary-porous bodies.

The second part is devoted to the solution of the system of differential equations of heat and mass transfer under different boundary conditions. The solutions obtained can be used to determine the optimal conditions for the kinetics of a process; they enable us to find, for bodies of different

configurations, the distribution of the transfer potentials, taking into account their mutual influence (the non-stationary fields of temperature and transfer potential of the associated matter). The solution of these problems is accompanied by an analysis which aims at revealing the physical essentials of the process, and is illustrated so far as possible by figures and tables. A wide use is made of the theory of similarity, and thus the majority of the solutions of the equations of heat and mass exchange are presented in the criterial form. The mutual relations of the criteria of similarity are established.

It is generally understood how serious are the difficulties encountered by engineers in applying the methods of mathematical physics to the solution of engineering problems. Hence in this section special attention is paid to the technique of calculation in the solution of concrete problems, and rigorous mathematical investigation is omitted. With this end in view, we examine in detail the process of solution of a few of the simplest problems in the theory of heat and mass transfer, with step-by-step transformations which can be mastered by the engineer.

In the last section solutions are given for a number of problems in which the transfer of material takes place under the influence of thermodynamical forces, including the total pressure gradient. These mixed problems of the molecular-molar transfer of a mass of associated matter are of practical interest for the case of the thermal treatment of capillary-porous bodies.

The book contains a large number of original tables of fundamental solutions and parametric coefficients. Thus it is of interest not only to physicists and students of energetics, but also to a wider circle of readers. Thanks to its detailed and systematic exposition it can be understood by research workers in engineering, who will be able to apply analytical methods of investigation successfully to the most important problems of contemporary technology.

processes for any substance on the basis of the transfer of entropy, which uniquely characterizes the complex and interrelated molecular transfer processes.

In contrast to the differential equation of the transfer of energy, the differential equation of the transfer of entropy contains a source of entropy, corresponding to the increase of entropy in irreversible processes. This source of entropy is a function of the coordinates and the time. Hence the solution of differential equations of the heat conduction type with variable sources and sinks is of material importance for the development of the analytical theory of the thermodynamics of irreversible processes.

Moreover, the fundamental relation of the thermodynamical theory of transfer (the product of the absolute temperature by the rate of change of the volume concentration of entropy is equal to the sum of the products of the fluxes by the motive forces of the transfer), forms the initial relation for determining the transfer potentials of heat, matter, and energy of ordered motion.

§ 1. Transfer Potentials

Let us consider first of all the transfer potentials. The fundamental axiom of the theory of potentials is the following relation: in a condition of thermodynamic equilibrium the potentials of a system of bodies, or the potentials of different parts of the same body, are identical:

$$\Pi_1 = \Pi_2 . \tag{1.1}$$

The transfer of the respective entities (heat, electricity or matter) proceeds in the direction from the higher potential to the lower: $\Pi_1 > \Pi_2$.

The potential gradient is called the thermodynamic motive force; for example, the gradient of the electric potential is equal to the intensity of the electric field.

The quantitative measure of the transfer of energy dE is equal to the product of the potential by the change of the generalized

coordinate dK, that is, the transfer potential must satisfy the relation

$$dE = \Pi\, dK\,. \tag{1.2}$$

We will append a few examples. In the expansion of a gas, for an increase of volume dV, the work performed in the expansion against the external pressure p is equal to:

$$dW = p dV. \tag{1.3}$$

Hence the generalized coordinate for the case of the performance of mechanical work is the volume ($K = V$), and the potential is the pressure ($\Pi = p$).

The work dW of charging an electrical conductor at potential q ($\Pi = q$) in an electrostatic field is equal to the product of the potential q by the increase of the electric charge dq:

$$dW = q\, dq\,. \tag{1.4}$$

The generalized coordinate in this case is the magnitude of the electric charge ($K = q$). In heat exchange processes the quantity of heat transferred, dQ, is equal to the product of the temperature T (the transfer potential of heat energy) by the increase of entropy dS (the generalized coordinate is the entropy):

$$dQ = T dS\,. \tag{1.5}$$

In the transfer of energy by mass exchange the generalized coordinate is the mass of matter r, measured in moles, and the transfer potential is the corresponding chemical potential μ, which depends on the thermomechanical relations of the given system to the surrounding medium.

Let us consider this in more detail. The equation of the first law of thermodynamics for open systems (i.e., systems in which there is mass exchange) is the Gibbs equation

$$dQ = T dS = dE + p dV - \sum_k \mu_k\, d r_k\,, \tag{1.6}$$

where the subscript k denotes the number of material components taking part in the mass exchange.

For a system having mechanical and thermal relations with the surrounding medium, the following four relations are possible;†

(a) the isochoro-isentropic relation $V = \text{const.}$, $S = \text{const.}$ ($dV = 0$; $dS = 0$), whence we obtain for the transfer potential of mass

$$\mu = \left(\frac{\partial E}{\partial v}\right)_{v,s}; \tag{1.7}$$

(b) the isobaro-isentropic relation $p = \text{const.}$, $S = \text{const.}$ ($pdV = dpV$; $dS = 0$).

Let us denote the enthalpy of the system by $H (H = E + pV)$. Then from (1.6) we have

$$\mu = \left(\frac{\partial H}{\partial v}\right)_{p,s}; \tag{1.8}$$

(c) the isochoro-isothermal relation $V = \text{const.}$, $T = \text{const.}$ ($dV = 0$; $TdS = dTS$). In this case we have

$$\mu = \left(\frac{\partial F}{\partial v}\right)_{v,T}, \tag{1.9}$$

where F is the free energy, $F = E - TS$;

(d) the isobaro-isothermal relation $p = \text{const.}$, $T = \text{const.}$ ($pdV = dpV$; $TdS = dTS$).

Let us denote the free enthalpy by Φ, i.e.

$$\Phi = H - TS = E + pV - TS,$$

then, from equation (1.6), we have

$$\mu = \left(\frac{\partial \Phi}{\partial v}\right)_{p,T}. \tag{1.10}$$

Hence the transfer potential of mass depends on the conditions of the thermomechanical relationship between the system and the surrounding medium; it is equal to the derivative of the corresponding characteristic function ψ with respect to the mass:

$$\mu = \frac{\partial \psi}{\partial v}. \tag{1.11}$$

† It is assumed that all the v_k except one are constant.

In the first case the characteristic function is the internal energy ($\psi = E$), and in the second case the enthalpy ($\psi = H$), and so forth.

§ 2. The Fundamental Relations of the Thermodynamics of Irreversible Processes

Two principles lie at the foundation of the thermodynamics of irreversible processes: the linear law and Onsager's reciprocal relations. According to the linear law, the velocity v of the approach of the system to a condition of equilibrium is proportional to the thermodynamic motive force X, which can be expressed by the potential gradient (the specific flux of electricity is proportional to the potential gradient or the intensity of the field, the heat flux in a process of heat exchange by conduction is proportional to the temperature gradient, the diffusion flux is proportional to the concentration gradient, and so on).

The linear law is a generalization of the empirical relations established for systems whose condition is near to that of equilibrium. Hence the linear law is an experimental law and has no rigorous theoretical basis. It should be noted, however, that the linear law can be deduced from the statistical theory of fluctuations, if the condition of the system is near to equilibrium.

In the presence of several motive forces the velocity v_i with which the system approaches equilibrium, according to the linear law, can be expressed by the sum:

$$v_i = \sum_j L_{ij} X_j. \tag{2.1}$$

The terms whose coefficients are elements of the leading diagonal of the matrix represent the direct effect corresponding to the fundamental motive force, and the remaining terms represent the influence of the superimposed phenomena.

Individual applications of the relation (2.1) have long been known, but the general formulation of the linear law was

first given in the thermodynamics of irreversible processes. The thermodynamic motive force is defined on the basis of the fundamental relation of the thermodynamics of irreversible processes: the product of the rate of increase of the entropy, $\dot{S}$, by the absolute temperature is equal to the sum of the products of the motive forces by the rate of approach of the system to equilibrium:

$$\dot{S}T = \sum_i X_i v_i . \tag{2.2}$$

Making use of the relation (2.1) we obtain

$$\dot{S}T = \sum_{i,j} L_{ij} X_i X_j . \tag{2.3}$$

Since $\dot{S}$ *is greater than zero,* the sum on the right-hand side of (2.3) is an essentially positive quantity.

The second principle of the thermodynamics of irreversible processes is the reciprocity relation

$$L_{ij} = L_{ji}, \tag{2.4}$$

which states that the cross kinetic coefficients L are equal. This relation was proved by Onsager on the basis of the most general propositions of statistical mechanics.

The principle of reciprocity is the reflection of microscopic reversibility, i.e. that in conditions of equilibrium any molecular process and its reverse process proceed on the average with identical rates.

On these foundations the formal theory of the rate of irreversible processes was worked out, and was successfully applied to the phenomena of transfer and of chemical kinetics. Here the fundamental transfer equation is the differential equation of the transfer of entropy, which, in comparison with the Umov differential equation of the movement of energy, possesses one extra term expressing the existence of a positive source of entropy (the increase of entropy in irreversible processes). The laws of classical thermodynamics can be deduced by the methods of statistical mechanics,

for which reason some investigators are inclined to regard classical thermodynamics as a branch of statistical mechanics. The molecular-kinetic theory of heat affords not only a graphic physical explanation of the fundamental relations of thermodynamics, but also a certain complement to the results of classical thermodynamics. The successful combination of these methods makes possible a deeper and more detailed understanding of a number of physical and physico-chemical phenomena.

§ 3. Onsager's System of Linear Equations

We shall denote the specific fluxes (the quantities of the entities in question transferred across unit area of an equipotential surface in unit time) by I, the thermodynamic motive forces by X, and finally, the kinetic coefficients by L. Under these conditions the molecular transfer of electrical energy, heat or matter in a homogeneous body is described by the following system of linear equations:

$$\left.\begin{aligned} I_1 &= L_{11} X_1 + L_{12} X_2 + L_{13} X_3 \\ I_2 &= L_{21} X_1 + L_{22} X_2 + L_{23} X_3 \\ I_3 &= L_{31} X_1 + L_{32} X_2 + K_{33} X_3 \end{aligned}\right\}. \qquad (3.1)$$

Here Onsager's relation of reciprocity applies:

$$L_{12} = L_{21}\,;\; L_{13} = L_{31}\,;\; L_{23} = L_{32}\,. \qquad (3.2)$$

The thermodynamic motive force of the heat flux is the ratio of the temperature gradient to the temperature:

$$X_1 = X_q = -\frac{\nabla T}{T}\,. \qquad (3.3)$$

Analogous relations apply to the motive forces of matter and electricity:

$$X_2 = X_m = -T\nabla\left(\frac{\mu}{T}\right). \qquad (3.4)$$

where μ is the chemical potential (the derivative of the thermo-

dynamic function with respect to the mass of transferred matter in an isochoro-isothermal relation,

$$X_3 = X_c = -\nabla\varphi, \tag{3.5}$$

where φ is the electric potential.

It has been shown by the work of A. V. Luikov and A. P. Porkhaev that hydrodynamical motive forces influence the transfer of heat and matter in the boundary layer when a moist body is surrounded by a current of warm gas. In this case it is appropriate to regard internal friction or viscosity as a process of molecular transfer of the energy of orderly motion. The viscosity of a liquid or gas is usually regarded as the transfer of a quantity of orderly motion. In this case Newton's law of viscous flow cannot be compared with the laws of heat conduction and diffusion in the form of Onsager's system of linear equations, since the motive forces of the transfer of heat, matter and quantity of motion are tensors of different ranks. On the basis of Curie's theorem, only tensors of the same rank or those whose ranks differ by an even number can be included in Onsager's system of equations. This condition is not fulfilled by the motive forces of heat and of quantity of orderly motion. However, if we regard viscosity as a molecular transfer of energy of orderly motion, then the contradiction to Curie's theorem falls to the ground since the motive forces of the transfer of heat, matter and energy of orderly motion are tensors of one and the same rank.

The motive force of the molecular transfer of kinetic energy of orderly progressive motion is the gradient of the kinetic energy of progressive motion of unit mass of the liquid, i.e.

$$X_4 = X_e = -\nabla\left(\frac{\gamma w^2}{2}\right), \tag{3.6}$$

where γ and w are respectively the density and linear velocity of motion of the liquid or gas.

In this case a further equation must be added to the system of linear equations (3.1):

$$I_4 = L_{41} X_1 + L_{42} X_2 + L_{43} X_3 + L_{44} X_4, \qquad (3.7)$$

where I_4 is the specific flux of kinetic energy of orderly motion transferred as a result of the chaotic molecular motion.

It follows form the relations (3.3)—(3.6) that the thermodynamic motive force corresponding to any form of transfer is directly proportional to the potential gradient of that form of transfer. The minus sign is necessary owing to the fundamental axiom of transfer: that transfer proceeds from higher potential to lower. If we denote the potential of transfer i by Π_i, then we can write

$$X_i = -K_i \nabla \Pi_i. \qquad (3.8)$$

The coefficient of proportionality K_i depends on the choice of fluxes and forces, and in a particular case it may be equal to unity. In the thermodynamics of irreversible processes it is proved that Onsager's reciprocity relation is valid for any choice of fluxes and forces, i.e. that the matrices L_{ik} possess invariant symmetry.

Thus in the general case, molecular transfer is represented by the following system of Onsager linear equations:

$$\left.\begin{aligned} I_1 &= \sum_i L_{1i} X_i \\ I_2 &= \sum_i L_{2i} X_i \\ I_3 &= \sum_i L_{3i} X_i \\ &\cdots\cdots \\ I_i &= \sum_i L_{ji} X_i \end{aligned}\right\}. \qquad (3.9)$$

According to the type of transfer, we can confine ourselves to a few thermodynamic forces.

§ 4. Differential Equations of Heat and Mass Transfer in Binary Gaseous Mixtures

The differential equation of mass transfer is deduced on the basis of the law of conservation of matter, and here the methods of determining the specific flux of the k-component, I_k, in relation to the speed of motion of the gaseous mixture, w, is very important.

If the flux of the k-component I_k is determined relatively to the centre of mass of the whole mixture, M, moving with velocity w, by the relation

$$I_k = \omega_k(w_k - w) \tag{4.1}$$

where ω_k is the concentration of component k, equal to the ratio of the mass of the component, M_k, to the volume of the mixture, V ($\omega_k = M_k/V$); and w_k is the velocity of motion of the component, then we have the equality

$$\sum_{k=1}^{n} I_k = 0\,, \tag{4.2}$$

so that the velocity of the centre of gravity of the mass

$$w = \frac{1}{\omega}\sum_k \omega_k = w_k \tag{4.3}$$

where ω is the total concentration or density of the mixture,

$$\omega = \sum_k \omega_k = \frac{1}{V}\sum_k M_k = \frac{M}{V}\,. \tag{4.4}$$

For a binary mixture ($k = 1, 2$) it follows from (4.2) that

$$I_1 = -I_2 \tag{4.5}$$

i.e. the fluxes of matter by mass (kg/m^2.hr) in mutual diffusion are equal and directed oppositely. Hence Onsager's system of linear equations for isothermal mutual diffusion

$$\begin{aligned} I_1 &= L_{11}X_1 + L_{12}X_2 \\ I_2 &= L_{21}X_1 + L_{22}X_2 \end{aligned} \tag{4.6}$$

degenerate into the single equation

$$J_1 = -J_2 = L_{11}(X_1 - X_2) = -L_{12}(X_1 - X_2)\,. \quad (4.7)$$

where X_k is the thermodynamical force of mass transfer; L_{ik} are kinetic coefficients ($k = 1, 2$; $i = 1, 2$).

From equation (4.7) it is clear that in the present case Onsager's relations do not apply, since the mutual isothermal diffusion of two components is an isolated phenomenon and there are no super-imposed effects.

On the basis of Prigogine's theorem [3], for mechanical equilibrium ($p = \text{const.}$) the increase in entropy does not depend on the choice of the mean velocity w, by means of which the flux I_k is determined from the relation (4.1).

Let us suppose that the arbitrary mean velocity is

$$w_c = \frac{\sum_k' G_k w_k}{\sum_k' G_k} \quad (4.8)$$

where G_k is the weight function. Then

$$I_{ck} = \omega_k(w_k - w_c)\,. \quad (4.9)$$

From the relations (4.8) and (4.9) we obtain the equality

$$\sum_k \frac{G_k}{\omega_k} I_{ck} = 0\,. \quad (4.10)$$

The formulae considered above (4.1)–(4.5) correspond to the case where the function G_k is equal to the concentration of the k-component ($G_k = \omega_k$, $w_c = w$). Then we can write for the isothermal mutual diffusion

$$I_1 = -I_2 = -\omega D_c \nabla \omega_{10} \quad (4.11)$$

where ($\omega_{10} = \omega_1/\omega$) is the relative concentration or specific mass content of component 1 ($\omega_{10} = M_1/M$); and D_c is the coefficient of mutual diffusion. The mutual diffusion defined by the relations (4.9)–(4.11) is called concentrational or "gravitational" diffusion, following de Groot's terminology [3].

If the function G_k is put equal to the molar concentration (the number of moles of component k in unit volume), i.e. $G_k = \omega_k/\mu_k$, where μ_k is the molecular weight, then we obtain from relations (4.8)–(4.10)

$$\sum_k \frac{I_{mk}}{\mu_k} = 0 . \tag{4.12}$$

Here we assume that $w_c = w_m$, $I_{ck} = I_{mk}$. Then for mutual diffusion in a binary gaseous mixture we can write:

$$\frac{I_1}{\mu_1} = -\frac{I_2}{\mu_2} = -\nu D_m \nabla \nu_{10} \tag{4.13}$$

where ν_{10} is the relative molar concentration ($\nu_{10} = \nu_1/\nu$).

In this case we have equality of the molar fluxes of matter (mol/m^2. hr), and the mutual diffusion is said to be molecular.

By the use of the second theorem of Prigogine [3] we can establish a connection between the coefficients of concentrational (D_c) and molecular diffusion (D_m).

The differential equation representing the law of conservation of mass for the k-component may be written:

$$\frac{\partial \omega_k}{\partial \tau} = -\operatorname{div} \omega_k w_k + j_{vk} \tag{4.14}$$

where j_{vk} is the volume power of the source or sink of the k-component due to phase changes or chemical reactions.

If the velocity of motion of the mixture, w, is defined with respect to the centre of gravity of the mass (see relation 4.3), then, summing equation (4.14) for all components of the mixture, we have

$$\frac{\partial \omega}{\partial \tau} = -\operatorname{div} \omega w \tag{4.15}$$

so that $\Sigma j_{vk} = 0$.

The equality (4.15) thus obtained represents the equation of continuity and can be expressed by the total derivative in the form

$$\frac{d\omega}{d\tau} = -\omega \operatorname{div} w \tag{4.16}$$

where $\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + w \operatorname{grad}$.

In the case of concentrational diffusion (see relation 4.9) the differential equation (4.14) can be written:

$$\frac{d\omega_k}{d\tau} = -\omega_k \operatorname{div} w - \operatorname{div} I_k + J_{rk}. \qquad (4.17)$$

Making use of relation (4.16), the last equation can be transformed into

$$\omega \frac{d\omega_{k0}}{d\tau} = -\operatorname{div} I_k + j_{rk} \qquad (4.18)$$

which will also apply in the case of a mixture of variable density ($\omega_{k0} = \omega_k/\omega$).

In the derivation of the equations of transfer by the methods of the molecular-kinetic theory the diffusion flux J_k is defined as the flux of molecules in unit time across unit surface, i.e. $J_k = I_{nk}/m_k$, where m_k is the mass of a molecule of the k-component. In this case the function G_k is equal to the number of molecules of the k-component per unit volume ($G_k = n_k$).

Then for mutual molecular diffusion in a binary mixture ($k = 1, 2$), according to relation (4.10), taking into account the equality $\omega_k = m_k n_k$, we can write:

$$\frac{I_{n1}}{m_1} = -\frac{I_{n2}}{m_2}; \; J_1 = -J_2, \qquad (4.19)$$

i.e. we have equality of the fluxes of molecules, which is equivalent to equation (4.13).

For a binary mixture at constant pressure and variable temperature, in the absence of external forces, the fundamental transfer equation has the form [1]

$$w_1 - w_2 = -\frac{1}{n_{10} n_{20}} \left(D_{12} \nabla n_{10} + \frac{D_T}{T} \nabla T \right), \qquad (4.20)$$

where n_{10}, n_{20} are the relative proportions of the molecules ($n_{10} + n_{20} = 1$),

$$n_{10} = \frac{n_1}{n}, \; n_{20} = \frac{n_2}{n};$$

D_{12} and D_T are respectively the coefficients of mutual molecular diffusion and of thermal diffusion.

The flux J_1, according to the relation

$$J_1 = n_1(w_1 - w), \tag{4.21}$$

where the velocity of movement of the mixture w is defined on the basis of (4.8) by the following formula:

$$w = \frac{n_1 w_1 + n_2 w_2}{n}. \tag{4.22}$$

Hence we shall have

$$J_1 = n_1 n_{20}(w_1 - w_2) = - n D_{12}\left(\nabla n_{10} + \frac{k_T}{T}\nabla T\right), \tag{4.23}$$

where $k_T = D_T/D_{12}$ is the ratio of the coefficient of thermal diffusion to the coefficient of molecular diffusion.

The differential equation of transfer for molecules of the k-component in the absence of sources and sinks of matter ($j_{vk} = 0$) has the form

$$\frac{\partial n_k}{\partial \tau} = - \operatorname{div} n_k w_k. \tag{4.24}$$

Summing (4.24) for all components we have:

$$\frac{dn}{d\tau} = - n \operatorname{div} w. \tag{4.25}$$

If we make use of the equality (4.21) the differential equation (4.24) can be written:

$$\frac{\partial n_{k0}}{\partial \tau} = - \operatorname{div} J_k - \operatorname{div} n_k w. \tag{4.26}$$

This relation reduces to

$$n \frac{dn_{k0}}{d\tau} = - \operatorname{div} J_k, \tag{4.27}$$

by the use of (4.25).

For a binary mixture the flux J_1 is defined by formula (4.23), hence the differential equation of the transfer of molecules (4.27) will have the form:

$$\frac{dn_{10}}{d\tau} = D_{12}\left(\nabla^2 n_{10} + \frac{k_T}{T}\nabla^2 T\right). \qquad (4.28)$$

In the transition from (4.27) to (4.28) it is assumed that the quantities nD_{12} and k_T/T change very little with the coordinates. The relative content of molecules n_{10} is equal to the relative partial pressure p_{10} ($p_{10} = p_1/p$) or the relative molar concentration v_{10}, i.e.

$$n_{10} = p_{10} = v_{10}. \qquad (4.29)$$

Hence equation (4.28) is the differential equation of mass transfer of matter, since n_{10} can be replaced by the partial pressure p_1, and instead of the coefficient k_T we can write pk_T. The flux of enthalpy I_h relative to the immovable surface in this system of coordinates, in which the velocities of the molecules are denoted by the relation (4.22), will be equal [1] to:

$$I_h = -\lambda\nabla t + k_T p\frac{n}{n_1 n_2}J_1 + c_p\omega t w, \qquad (4.30)$$

where λ is the coefficient of thermal conductivity and c_p is the specific heat of the mixture under constant pressure.

The first term of formula (4.30) corresponds to the transfer of heat by conduction, whilst the third term represents the transfer of heat by the motion of the mixture (convection). The second term defines the magnitude of the transfer of heat by diffusion. It is equal to the specific flux of diffusional conductivity $(I_q)_{dif}$.

$$(I_q)_{dif} = \frac{k_T p}{\omega_1 n_{20}} I_1 = \frac{k_T p}{n_{20}} u_{dif}, \qquad (4.31)$$

where u_{dif} is the linear rate of the diffusional transfer of matter ($u_{dif} = I_1/\omega_1$).

The differential equation of the transfer of heat is obtained on the basis of the law of conservation of energy, using relation (4.27), i.e.

$$c_p\omega\frac{\partial t}{\partial\tau} = -\operatorname{div} I_h = -c_p\omega w\nabla t + \lambda\nabla^2 t + bp\frac{dn_{10}}{d\tau}, \qquad (4.32)$$

where $b = k_T/n_{10}n_{20}$ is the thermodiffusional constant.

In its usual form expression (4.32) is written:

$$\frac{dt}{d\tau} = a\nabla^2 t + \frac{bp}{c_p\omega}\frac{dn_{10}}{d\tau}, \qquad (4.33)$$

where a is the coefficient of temperature conductivity of the mixture, $a = \lambda/c_p\omega$.

Differential equations (4.28) and (4.33) are a mutually connected system of equations of heat and mass transfer in a moving binary mixture, which completely describes the molecular and molar (convective) transfer of heat and mass, taking account of the effects of superposition. This system may be represented in non-dimensional form:

$$\frac{dt^*}{d\mathrm{Fo}} = \nabla^{*2} t^* + \mathrm{Du}\frac{dn_{10}^*}{d\mathrm{Fo}}, \qquad (4.34)$$

$$\frac{dn_{10}^*}{d\mathrm{Fo}} = \mathrm{Lu}\,(\nabla^{*2} n_{10}^* + \mathrm{So}\,\nabla^{*2} t^*) \qquad (4.35)$$

where $t^* = t/t_c$, $n_{10}^* = n_{10}/n_{10}^0$, and t_c and n_{10}^0 are respectively the temperature of the medium and the relative initial content of molecules or the relative initial partial pressure ($n_{10}^* = p_{10}^* = p_1^0/p$),

$$\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}},$$

$$x^* = x/X,\ y^* = y/Y,\ z^* = z/Z. \qquad (4.36)$$

From the system of differential equations of heat and mass transfer we obtain the well-known criteria of similarity Fo, Pe and Lu

$$\mathrm{Fo} = \frac{a\tau}{l^2},\quad \mathrm{Pe} = \frac{wl}{a},\quad \mathrm{Lu} = \frac{D_{12}}{a}, \qquad (4.37)$$

where l is the characteristic dimension of the body. Two other criteria, So and Du, enter into this system of equations. The former is called the Soret criterion:

$$\mathrm{So} = bn_{20}^0 = b(1 - n_{10}^0). \qquad (4.38)$$

It characterizes the thermodiffusional effect defined by the thermodiffusional constant.

The thermodiffusional constant b depends on the law of mutual action of similar and dissimilar molecules and on the temperature. It can be calculated theoretically only for isotropic mixtures. Usually the constant b is determined from the experimental data of thermodiffusion or the diffusional thermo-effect.

For binary gas mixtures with equal content of the components ($n_{10}^0 = n_{10}^0 = 0{\cdot}5$) the Soret constant is equal to half the thermodiffusional constant.

The second criterion is that of Dufour:

$$\mathrm{Du} = \frac{R}{C_p} b n_{10}^0 , \tag{4.39}$$

where R and C_p are respectively the universal gas constant and the molar heat capacity of the mixture at constant pressure ($C_p = \mu c_p$).

In order to explain its physical significance we will write the Dufour criterion in another form:

$$\mathrm{Du} = \frac{k_T p}{c_p \omega t_c n_{10}} = \frac{(I_q)_{\mathrm{dif}}}{c_p \omega t_p u_{\mathrm{dif}}} . \tag{4.40}$$

Hence the criterion Du is equal to the ratio of the flux of energy (heat) of the diffusional heat conduction to a certain conventional convective flux of heat, which would take place if the velocity of motion of the mixture were equal to the linear velocity of diffusion ($w = u_{\mathrm{dif}}$). In other words, the *Dufour criterion is a measure of the diffusional transfer* of heat in comparison with the convective transfer under conditions where the diffusional and convective linear transfer rates are equal.

It is obvious from its physical meaning that the Dufour criterion is always less than unity ($0 < \mathrm{Du} < 1$). It depends on the initial concentration, the thermodiffusional constant b and the molar heat capacity. Since the latter are determined by the temperature and the initial concentration, the criterion Du is a function of the temperature and the concentration or partial pressure.

§ 5. Transfer Equations of Heat and Mass in Disperse Media in Case of Phase Change

Let us consider the transfer of heat and mass in a disperse medium in the presence of phase changes.

The disperse medium contains matter in the vapour (subscript 1), liquid (subscript 2), and solid (subscript 3) forms, i.e. the disperse medium has bound matter in three phases. Subscript 0 denotes the disperse medium itself, without associated matter (the skeletal disperse medium).

The volume concentration ω (mass of matter in unit volume of the medium) of bound matter is equal to the product of the specific mass content u (the mass of associated matter per unit mass of the skeletal disperse medium) by the density γ_0 of the medium (mass of skeletal disperse medium per unit volume):

$$\omega = u\gamma_0 ; \quad \omega = \sum_{i=1}^{3} \omega_i ; \quad u = \sum_{i=1}^{3} u_i . \tag{5.1}$$

A local change in the associated matter du_i in any phase i may occur as the result of a phase change du_{ij} or by a transfer du_{if},

$$du_i = \sum_j du_{ij} + du_{if} . \tag{5.2}$$

Here the equality (5.3) holds, by virtue of the law of conservation of mass in transition from phase i to phase j or inversely

$$du_{ij} + du_{ji} = 0 ; \quad du_{ij} = - du_{ji} . \tag{5.3}$$

Let us delimit a certain sufficiently small volume, bounded by a closed surface (the control surface). A change in the mass content in phase i takes place on account of transfer and the action of sources or sinks of matter occasioned by phase changes. Hence the differential equation of transfer of matter in any phase may be written

$$\gamma_0 \frac{\partial u_i}{\partial \tau} = - \operatorname{div} I_{mi} + \sum_j \gamma_0 \frac{\partial u_{ji}}{\partial \tau} \tag{5.4}$$

where I_{mi} is the density of the flux of i-phase matter; $\partial u_{ji}/\partial \tau$

is the volume power of the source of i-phase matter (transformation of matter from phase j to phase i in unit time per unit volume). Summing equation (5.4) with respect to i ($i = 1, 2, 3$) we obtain

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\operatorname{div} \sum_i I_{mi} \tag{5.5}$$

and since phase transformations within the delimited volume do not change the total mass content of bound matter within the volume,

$$\sum_i \sum_j \gamma_0 \frac{\partial u_{ji}}{\partial \tau} = 0 . \tag{5.6}$$

Hence a change in the total mass content in an infinitely small volume takes place only by the transfer of matter in the different phases.

If the internal energy E of volume V of the system with total mass M is calculated per unit mass (specific values of energy and volume), the expression of the first law of thermodynamics will be the same for a closed and for an open system

$$dq = de + pdv \tag{5.7}$$

where $v = V/M$ is the specific volume, $e = E/M$ is the specific energy, and p is the pressure.

For the whole quantities Q, E, V, referring to the total mass of an open system, this relation will have the form

$$dQ = dE + pdV - hdM \tag{5.8}$$

where h is the specific enthalpy,

$$h = e + pv , \tag{5.9}$$

and dQ is the quantity of heat transferred to the system ($dQ = Mdq$).

Let us denote the density of the fluxes of matter, heat, energy and enthalpy respectively by I_m, I_q, I_e and I_h:

$$I_m = \frac{dM}{A\,d\tau}, \quad I_q = \frac{dQ}{A\,d\tau}, \quad I_e = \frac{dE}{A\,d\tau}, \quad I_h = \frac{dH}{A\,d\tau}, \tag{5.10}$$

where A is an equipotential surface, i.e. a surface having everywhere the same transfer potential, and H is the total enthalpy ($H = hM$). For constant volume processes ($V =$ $=$ const.) of heat and mass transfer on the basis of formula (5.8) we have

$$I_e = I_q + hI_m . \tag{5.11}$$

For processes under constant pressure ($p =$ const.) we have from formula (5.8)

$$I_h = I_q + hI_m . \tag{5.12}$$

Let us consider the case of a constant pressure relation of the disperse system with the surrounding medium. Then the local rate of change of the enthalpy h of the system (disperse medium and bound matter) will be equal to the divergence of the flux density of the enthalpy

$$\frac{\partial}{\partial \tau}\left(\gamma_0 h_0 + \sum_{i=1}^{3} \gamma_0 h_0 u_i\right) = -\operatorname{div}\left(I_q + \sum_{i=1}^{3} h_i I_{mi}\right). \tag{5.13}$$

Let us denote the specific heat at constant pressure by c:

$$c_i = \frac{dh_i}{dt}\,; \quad c_0 = \frac{dh_0}{dt}\,. \tag{5.14}$$

Making use of the differential equation of the transfer of matter (5.4) we obtain from the relation (5.13) the differential equation of the transfer of heat in the following form:

$$c\gamma_0 \frac{\partial t}{\partial \tau} = -\operatorname{div} I_q + \sum_i \sum_j h_i \gamma_0 \frac{\partial u_{ji}}{\partial \tau} - \sum_i c_i I_{mi} \nabla t \tag{5.15}$$

where c is the reduced specific heat of the system considered:

$$c = c_0 + \sum_{i=1}^{3} c_i u_i . \tag{5.16}$$

The first term on the right-hand side of equation (5.15) represents the molecular transfer of heat. It includes the transfer of heat by conduction and the transfer by the diffusion of matter (diffusion heat conduction), i.e.

$$I_q = -\lambda \nabla t + (I_q)_{\text{dif}} . \tag{5.17}$$

Hence, from relation (4.30), the third term on the right-hand side of (5.15) represents the transfer of heat by macroscopic motion of matter (convective transfer), i.e.

$$c_l I_{ml} \nabla t = c_l \gamma_l w_l \nabla t \qquad (5.18)$$

where w_l is the linear velocity of matter ($w_l = I_{ml}/\gamma_l$). The second term on the right-hand side of (5.15) represents a source of heat due to phase transformations. The differential equations (5.4), (5.5) and (5.15) describe the transfer of heat and mass in the disperse system under any law of transfer of matter. Let us consider the particular case of two phases of associated matter (vapour and liquid, $i = 1, 2$), in a region of positive temperature on the Centigrade scale ($t = T - 273 > 0$). In this case the differential equations of transfer of matter (5.4) and (5.5) can be written

$$\gamma_0 \frac{\partial u_2}{\partial \tau} = -\operatorname{div} I_2 + \gamma_0 \frac{\partial u_e}{\partial \tau}, \qquad (5.19)$$

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\operatorname{div} (I_1 + I_2) \qquad (5.20)$$

where $\partial u_e/\partial \tau$ is a source or sink of liquid ($\partial u_e/\partial \tau = \partial u_{12}/\partial \tau = -\partial u_{21}/\partial \tau$), the subscript m being dropped for simplicity.

Here we must remember that under ordinary conditions (pressure of vapour and gas near to the barometric) the specific mass content of vapour u_1 in the pores of the medium is small compared with that of liquid u_2 ($u_1 \ll u_2$). Hence when the mass content of liquid differs from zero ($u_2 > 0$) we can regard the total mass content of bound matter u in the disperse medium as approximately equal to the mass content of liquid ($u \approx u_2$). The differential equation of heat transfer (5.15) in this case may be written:

$$c\gamma_0 \frac{\partial t}{\partial \tau} = -\operatorname{div} I_q + \varrho_e \gamma_0 \frac{\partial u_e}{\partial \tau} - \sum_{l=1}^{2} c_l \gamma_l w_l \nabla t, \qquad (5.21)$$

where ϱ_e is the specific heat of evaporation ($\varrho_e = \varrho_{12} = \varrho_{21}$)

equal to the difference of the enthalpy of vapour and liquid ($\varrho_{12} = h_1 - h_2$).

In the derivation of equation (5.21) we made use of the expression

$$h_1 \frac{\partial u_{12}}{\partial \tau} + h_2 \frac{\partial u_{21}}{\partial \tau} = (h_1 - h_2) \frac{\partial u_e}{\partial \tau} = \varrho_e \frac{\partial u_e}{\partial \tau} . \qquad (5.22)$$

This relation is valid when there is thermodynamical equilibrium of vapour and liquid in the pores of the disperse medium, corresponding to heat and mass transfer processes of low intensity. In the case of a three-phase system of bound matter ($i = 1, 2, 3$) we can introduce the following simplifications into the differential equations of heat and mass transfer.

The motion of the ice may be neglected ($I_3 = 0$). Then the transfer equation of matter for phase $i = 3$ assumes the form:

$$\gamma_0 \frac{\partial u_3}{\partial \tau} = \gamma_0 \frac{\partial u_{23}}{\partial \tau} + \gamma_0 \frac{\partial u_{13}}{\partial \tau} . \qquad (5.23)$$

It is usual to introduce the coefficient of icing, equal to the ratio of the mass of solid to the total mass of matter. In our symbols the coefficient of icing k is equal to

$$k = \frac{u_3}{u} = \frac{u_3}{u_2 + u_3} . \qquad (5.24)$$

If the disperse medium contains no ice ($u_3 = 0$) the coefficient $k = 0$. When the whole of the liquid has been changed into ice, $k = 1$. In most cases k is less than unity ($0 < k < 1$). From the relation (5.24) for $k = \text{const.}$ we have

$$\frac{\partial u_3}{\partial \tau} = \frac{k}{1 - k} \frac{\partial u_2}{\partial \tau} . \qquad (5.25)$$

Making use of this relation and also remembering that $u_2 \gg u_1$, $I_3 = 0$, the differential equation of transfer of matter may be written:

$$\frac{k \gamma_0}{1 - k} \frac{\partial u_2}{\partial \tau} = - \operatorname{div} (I_1 + I_2) . \qquad (5.26)$$

If we neglect sublimation processes ($du_{31} = 0$) and the reverse ($du_{13} = 0$) then the heat transfer equation will have the form:

$$c\gamma_0 \frac{\partial t}{\partial \tau} = -\operatorname{div} I_q + \varrho_e \gamma_0 \frac{\partial u_e}{\partial \tau} - \varrho_f \frac{k}{1-k} \gamma_0 \frac{\partial u_2}{\partial \tau} - \sum_{i=1}^{2} c_i \gamma_i w_i \nabla t, \tag{5.27}$$

where ϱ_f is the specific heat of evaporation of ice ($\varrho_f = h_3 - h_2$).

The first term on the right-hand side of equation (5.27) represents the molecular transfer of heat, the second term represents the heat source due to the condensation of vapour (or the heat sink due to the evaporation of liquid), the third term represents the heat sink due to the transition from solid to liquid, and the fourth term corresponds to the convective transfer of heat by currents of vapour and liquid.

In order that the system of differential equations of transfer may be complete it is necessary to express the fluxes of heat I_q and matter I_l by the gradients of the corresponding transfer potentials. This can be done on the basis of contemporary conceptions of the transfer of energy and matter by the methods of the thermodynamics of irreversible processes.

§ 6. The System of Differential Equations of the Transfer of Heat and Matter in Capillary-porous Bodies

Let us consider the transfer of heat, liquid and vapour in capillary-porous bodies in the presence of phase transformations (evaporation of liquid or condensation of vapour).

In the pores and capillaries of the porous body, vapour and air are present. This vapour–gas mixture can be regarded as a binary mixture (vapour and dry air). Then the transfer of vapour in the capillary-porous body can be described by the differential equation (4.28).

Under isothermal conditions ($T = \text{const.}$) the transfer of liquid in the capillary-porous body can be described

by the equation of isothermal mass-conductivity, according to which the density of the flux of liquid is directly proportional to the gradient of the specific mass-content, i.e.,

$$I_2 = -a_2 \gamma_0 \nabla u , \tag{6.1}$$

where a_2 is the coefficient of proportionality, whose dimensions are m^2/hr; $\nabla u = \nabla u_2$, since under ordinary conditions the content of vapour in the capillaries of the body is negligibly small in comparison with the content of liquid ($u_1 \ll u_2$), and we can consider $u_2 = u$.

The specific mass-content is not the transfer potential of the liquid, since at the boundary of two different porous bodies in a condition of equilibrium there is a discontinuity of mass-content.

The *mass-content* is the analogue of the enthalpy, and the coefficient a_2 is the analogue of the coefficient of temperature conductivity a_q (m^2/hr). This is easily seen from the following expression of Fourier's Law of Conductivity:

$$I_q = -\lambda_q \nabla t = -a_q \gamma_0 \nabla h , \tag{6.2}$$

where h is the specific enthalpy of the body.†

Hence the coefficient a_2 is called the coefficient of potential conductivity of transfer of liquid. The coefficient of temperature conductivity can be called the coefficient of potential conductivity of the transfer of heat, since the temperature is the potential of transfer of heat. The potential of transfer of liquid θ is a certain quantity proportional to the specific mass-content

$$u = \bar{c}_m \theta , \tag{6.3}$$

where $\bar{c}_m$ is the mean isothermal specific mass capacity of the capillary-porous body, its real value being determined by the relation:

$$c_m = \left(\frac{\partial u}{\partial \theta}\right)_T . \tag{6.4}$$

† The heat characteristics of the body are denoted by the subscript q.

The transfer potential θ is determined experimentally from the specific mass-content of a standard capillary-porous body in thermodynamic equilibrum with the moist porous body under investigation.

If we make use of the relation (6.4), the law of isothermal transfer of liquid can be written as follows:

$$I_2 = -a_2 c_m \gamma_0 \nabla \theta = -\lambda_m \nabla \theta . \tag{6.5}$$

The quantity $a_2 c_m \gamma_0$ can be called the coefficient of mass-conductivity λ_m, analogous to the coefficient of heat conductivity λ_q:

$$\lambda_m = a_2 c_m \gamma_0 ; \ \lambda_q = a_q c_q \gamma_0 . \tag{6.6}$$

In the case of non-isothermal conditions (T variable), the transfer of liquid proceeds under the influence of the gradient of transfer potential of matter $\nabla\theta$ and of transfer of heat ∇t, i.e.

$$I_2 = -\lambda_m \nabla \theta - \lambda_{mt} \nabla t , \tag{6.7}$$

where λ_{mt} is the coefficient of thermal mass-conductivity.

The transfer potential of matter θ is a function of the specific mass-content u and the temperature T. Hence we can write

$$\nabla \theta = \left(\frac{\partial \theta}{\partial u}\right)_T \nabla u + \left(\frac{\partial \theta}{\partial T}\right)_u \nabla T = \frac{1}{c_m} \nabla u + \left(\frac{\partial \theta}{\partial T}\right)_u \nabla t . \tag{6.8}$$

The quantity $(\partial\theta/\partial T)_u$ is the temperature coefficient of the transfer potential of matter.

Making use of the relation (6.8), we may write the formula for the transfer of liquid as follows

$$I_2 = -a_2 \gamma_0 \nabla u - a_2 \gamma_0 \delta_2 \nabla t , \tag{6.9}$$

where δ_2 is the thermal gradient coefficient for the transfer of liquid, analogous to the Soret coefficient

$$\delta_2 = c_m \left[\left(\frac{\partial \theta}{\partial T}\right)_u + \frac{\lambda_{mt}}{\lambda_m}\right] . \tag{6.10}$$

The coefficient δ_2 can be determined experimentally as the ratio of the gradients of mass-content and temperature

when $I_2 = 0$.

$$\delta_2 = -\frac{\nabla u}{\nabla t} = -\frac{\Delta u}{\Delta t}. \tag{6.11}$$

If the quantity λ_{mt}/λ_m is small in comparison with the quantity $(\partial\theta/\partial T)_u$, then the thermal gradient coefficient is equal to the product of the specific isothermal mass-capacity by the temperature coefficient of the potential of transfer of liquid,

$$\delta_2 = c_m \left(\frac{\partial\theta}{\partial T}\right)_u. \tag{6.12}$$

Hence the differential equation of the transfer of liquid can be written as follows:

$$\frac{\partial u}{\partial\tau} = \operatorname{div}\left(a_2 \nabla u + a_2 \delta_2 \nabla t\right) + \frac{\partial u_e}{\partial\tau}. \tag{6.13}$$

In the drying processes of capillary-porous bodies the source or sink of liquid $\partial u_e/\partial\tau$ can be expressed through the local rate of change of the mass-content of the body $\partial u/\partial\tau$ by means of the phase-change criterion ε:

$$\frac{\partial u_e}{\partial\tau} = \varepsilon \frac{\partial u}{\partial\tau}. \tag{6.14}$$

This is due to the fact that vapour and liquid do not enter a capillary-porous body from outside sources, but the formation of vapour in the moist body proceeds only by the evaporation of the liquid contained in it. The criterion of phase-change ε_{ij} of the bound matter from phase $i (i = 2)$ to phase j is determined by the ratio:

$$\varepsilon_{ij} = \frac{\partial u_{ij}}{\partial u_i}. \tag{6.15}$$

If the vectors of the fluxes of vapour I_1, and liquid I_2 are parallel, then the criterion of phase-change will be equal to

$$\varepsilon = \frac{I_1}{I_1 + I_2}. \tag{6.16}$$

Consequently the criterion ε is determined by the quantity of moving vapour in the moist body in relation to the total

current of vapour and liquid. When $\varepsilon = 1$ the transfer of matter proceeds only through the transfer of vapour ($I_2 = 0$). When $\varepsilon = 0$ the transfer of matter proceeds only in the form of liquid ($I_1 = 0$).

Thus, the equation of transfer of liquid (6.13) takes the form

$$\frac{\partial u}{\partial \tau} = \operatorname{div}(a_2 \nabla u + a_2 \delta_2 \nabla u) + \varepsilon \frac{\partial u}{\partial \tau}. \qquad (6.17)$$

This differential equation can be applied to the transfer of matter when there is chemical transformation.

In a hygroscopic region the equation of transfer of matter for a porous moist body (6.17) can be written in another form without introducing the criterion of phase-change ε. In this region there exists a definite relation between the partial pressure of the vapour P_1, the specific mass-content u and the temperature T. The form of the function $P_1 = f(u, T)$ is determined from the adsorption or desorption isotherm of the moist hygroscopic body. In this case we can make use of the differential equation of transfer of matter without sources or sinks, i.e.

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\operatorname{div}(I_1 + I_2). \qquad (6.18)$$

The magnitude of the flux of vapour I_1 according to formula (4.23) is determined thus

$$I_1 = -\frac{D \mu_1}{RT}\left(\nabla P_1 + \frac{k_T P}{T} \nabla t\right), \qquad (6.19)$$

where D is the coefficient of molecular diffusion of vapour in the capillary-porous body. This expression can be written as follows:

$$I_1 = -a_1 \gamma_0 \nabla u - a_1 \gamma_2 \delta_1 \nabla t, \qquad (6.20)$$

where a_1 is the coefficient of transfer potential conductivity of vapour in the porous body, equal to

$$a_1 = \frac{D \mu_1}{\gamma_c RT}\left(\frac{\partial P_1}{\partial u}\right)_T,$$

δ_1 is the thermal gradient of transfer of vapour, equal to

$$\delta_1 = \left[\left(\frac{\partial P_1}{\partial T}\right)_u + \frac{P k_T}{T}\right] \Big/ \left(\frac{\partial P_1}{\partial u}\right)_T .$$

Making use of the relations (6.9) and (6.20), equation (6.18) will have the form:

$$\frac{\partial u}{\partial \tau} = \operatorname{div}\left(a_m \nabla u + a_m \delta \nabla t\right), \tag{6.21}$$

where a_m is the coefficient of potential conductivity of the total transfer of vapour and liquid,

$$a_m = a_1 + a_2 , \tag{6.22}$$

δ is the thermal gradient coefficient for two-phase bound matter,

$$\delta = \frac{a_1 \delta_1 + a_2 \delta_2}{a_1 + a_2} . \tag{6.23}$$

It can be shown that for constant values of the transfer coefficients and of the criterion of phase-change the following relation holds

$$\delta = \varepsilon(\delta_1 - \delta_2)\,\lambda + \delta_2 ; \; a_m = \frac{a_2}{1 - \varepsilon} . \tag{6.24}$$

The equation (6.21) of transfer of matter (liquid and vapour) holds, as shown above, for a hygroscopic region. In a moist region of the body, when the mass-content is greater than the maximum hygroscopic mass-content, we must make use of equation (6.17). In the particular case where $\varepsilon = 1$ (transfer of liquid is absent), instead of the differential equation (6.21) we must make use of equation (4.28).

The differential equation of the transfer of heat is obtained from equation (5.21) by making use of the formula (6.2). It can be simplified on the basis of the following considerations. During the heating or drying of moist bodies the transfer of heat by liquid and vapour (the convective component of the transfer of heat) is very small in comparison with the conductive component, and with the sources of heat due

to phase-changes. Hence, the last term on the righthand side of equation (5.21) can be neglected if the Reynolds criterion Re for the moving vapour or liquid is less than unity, i.e.

$$\mathrm{Re} = \frac{w_l\, d}{\nu_l} < 1\,, \tag{6.25}$$

where ν_l is the kinematic viscosity of the vapour ($i = 1$) or liquid ($i = 2$), d is the equivalent diameter of the capillary of the porous body. Usually in drying processes the criterion Re for the total flux of vapour and liquid is less than 0·1. In view of what has been said the differential equation of heat transfer (5.21) is transformed into the equation of Fourier with a source or sink of heat due to phase-changes, i.e.

$$c_q\, \gamma_0 \frac{\partial t}{\partial \tau} = \mathrm{div}\,(\lambda_q \nabla t) + \varepsilon \varrho \gamma_0 \frac{\partial u}{\partial \tau}\,. \tag{6.26}$$

The subscript c is omitted for simplicity.

The system of differential equations of transfer of matter and heat (6.17) and (6.26) is a non-linear system since the coefficients of transfer λ_q and a_m and the thermodynamic characteristics δ and ε depend on the mass content and the temperature. However, if the calculation of the non-stationary heat and mass transfer is carried out by zones, in each of which the coefficients are taken as constant (mean value for each zone), then with a considerable simplification of the system of equations itself we obtain good agreement between calculation and experiment. Consequently, for a zonal system of calculation of the process of heat and mass exchange, the system of differential equations of transfer of heat and matter will take the form:

$$\frac{\partial u}{\partial \tau} = a_m \nabla^2 u + a_m \delta \nabla^2 t\,, \tag{6.27}$$

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \varepsilon \frac{\varrho}{c_q} \frac{\partial u}{\partial \tau}\,. \tag{6.27'}$$

The system of differential equations (6.27) and (6.27′) was first worked out by one of the authors of the present

monograph [14]. An analogous system of equations of transfer of heat and matter for the case of the absorption of moisture by a textile fibre was obtained by Crank [37]. The basis of this calculation was the assumption that the quantity of moisture absorbed u_e was a linear function of the concentration of moisture ω_1 and the temperature T, i.e.

$$u_e = \text{const.} + A\,\omega_1 + BT. \tag{6.28}$$

In our symbols, Crank's system of equations of transfer will have the form

$$\frac{\partial w_1}{\partial \tau} = \frac{k_1 D}{(k_1 + k_2)\,\Pi}\,\frac{\partial^2 \omega_1}{\partial x^2} + \frac{k_2 B}{A}\,a_q \frac{\partial^2 T}{\partial x^2}, \tag{6.29}$$

$$\frac{\partial T}{\partial \tau} = \frac{a_q}{k_1}\,\frac{\partial^2 T}{\partial x^2} + A\,\frac{P}{S_q}\,\frac{\partial \omega_1}{\partial x^2}, \tag{6.29'}$$

where Π is the porosity of the body, and k_1 and k_2 are constants equal to

$$k_1 = 1 + \frac{S}{C_q} B\,;\; k_2 = A\,\frac{1-\Pi}{\Pi \gamma_j},$$

and γ_j is the density of the threads of the textile material.

The system of differential equations (6.29) does not take into account the transfer of moisture in the liquid phase; it relates to the diffusion of vapour in the capillary-porous body in the presence of condensation or evaporation of liquid.

In the system of equations (6.27) the fundamental magnitudes are the temperature t and the specific mass-content u. The latter is not the transfer potential of matter. Hence, for a system of different bodies in contact, equation (6.27) is not applicable. In this case the transfer of heat and matter is described by the following system of transfer equations

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta_j \nabla^2 t, \tag{6.30}$$

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \varepsilon \varrho \frac{c_m}{c_q}\,\frac{\partial \theta}{\partial \tau}, \tag{6.30'}$$

where δ_j is the Soret coefficient for the moist body ($\delta_j = \lambda_{mt}/\lambda_m$). In the high-intensity heating of a moist capillary-porous body in a temperature region of 100°C, an additional thermodynamic motive force X_p, equal to the gradient of the total pressure ($X_p = -\nabla p$) appears. In the intense heating of a moist body a pressure appears within it higher than the surrounding barometric pressure because of the resistance of the skeleton of the porous body. This difference of the general pressure produces an additional transfer of liquid and vapour of the type of a filtrational movement through the capillary-porous body. Under these circumstances the flux of matter (vapour or liquid) will be described by the following expressions:

$$I_1 = -\lambda_p \nabla p, \tag{6.31}$$

$$I_2 = -\lambda_m \nabla \theta - \lambda_{mt} \nabla t, \tag{6.32}$$

where λ_p is the coefficient of mass conductivity of the filtrational motion of vapour. We neglect the molecular transfer of vapour (diffusion and thermo-diffusion) in comparison with the molar (filtrational) motion of the vapour. Thus in formula (6.31) the two first terms with gradients ∇p_{10} and ∇t are absent.

Making use of the relations (6.14) and (6.18), we can write the system of differential equations of heat and mass transfer thus:

$$c_q \gamma_0 \frac{\partial t}{\partial \tau} = \operatorname{div}(\lambda_q \nabla t) + \varepsilon \varrho \gamma_0 c_m \frac{\partial \theta}{\partial \tau} - \sum_{i=1}^{2} c_i w_i \gamma_i \nabla t, \tag{6.33}$$

$$c_m \gamma_0 \frac{\partial \theta}{\partial \tau} = \operatorname{div}(\lambda_m \nabla \theta + \lambda_m \delta_j \nabla t + \lambda_p \nabla p). \tag{6.34}$$

In order that the system of transfer equations may be a closed system, a third equation representing the change in the potential p of the filtrational motion of liquid and vapour is required. It can be obtained from equation (4.18) by taking into account the relations (6.14) and (6.31), and

also the porosity of the body

$$\Pi \frac{\partial \omega_1}{\partial \tau} = \operatorname{div}(\lambda_p \nabla p) - \varepsilon \gamma_0 c_m \frac{\partial \theta}{\partial \tau}, \tag{6.35}$$

where ω_1 is the concentration of vapour in the capillaries of the body. The minus sign is prefixed because with increasing mass-content of the liquid ($c_m \partial\theta/\partial\tau > 0$) the pressure of the vapour, and consequently also the concentration, ω_1, diminishes. In the general case ω_1 is a function of p and T, i.e. $\omega_1 = f(p, T)$. Then we can write:

$$\frac{\partial \omega_1}{\partial \tau} = A \frac{\partial p}{\partial \tau} + B \frac{\partial T}{\partial \tau}, \tag{6.36}$$

where A and B are respectively coefficients equal to

$$A = \left(\frac{\partial \omega_1}{\partial p}\right)_T; \; B = \left(\frac{\partial \omega_1}{\partial T}\right)_p.$$

Let us represent the vapour capacity of the capillary porous body by $c_p (c_p = \Pi A/\gamma_0)$. Then equation (6.35) can be written thus:

$$c_p \gamma_0 \frac{\partial p}{\partial \tau} = \operatorname{div}(\lambda_p \nabla p) - \varepsilon \gamma_0 c_m \frac{\partial \theta}{\partial \tau} - \frac{B\Pi}{c_p \gamma_0} \frac{\partial t}{\partial \tau}. \tag{6.37}$$

The system of differential equations of transfer (6.33), (6.34), and (6.37) completely describes the heat and mass transfer under the action of the three thermodynamic motive forces: ∇t, $\nabla \theta$, and ∇p.

Calculation shows that this system of equations of transfer can be simplified: the last members on the righthand side of equations (6.33) and (6.37) can be neglected, since they are small in comparison with the other terms of the equations.

Then for a zonal system of calculation we can write

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \varepsilon \varrho \frac{c_m}{c_q} \frac{\partial \theta}{\partial \tau}; \tag{6.38}$$

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta_j \nabla^2 t + a_p \nabla^2 p, \tag{6.38'}$$

$$\frac{\partial_p}{\partial \tau} = a_p \nabla^2 p - \varepsilon \frac{c_m}{c_p} \frac{\partial \theta}{\partial \tau}, \tag{6.38''}$$

where a_p is the coefficient of potential conductivity of the filtrational movement of the vapour, equal to

$$a_p = \frac{\lambda_p}{c_p \gamma_0} = \frac{\lambda_p}{\Pi A}. \tag{6.39}$$

To conclude this section we will consider the limiting conditions which consist of a combination of the boundary and initial conditions.

The boundary conditions reflect the law of the interaction of the surface of the body with the surrounding medium. This interaction consists of the exchange of heat and matter with the surrounding medium. Let us denote the density of the fluxes of heat and matter on the surface of the body respectively by q_q and q_m, i.e.

$$I_q = q_q(\tau)\,;\; I_m = q_m(\tau). \tag{6.40}$$

Then the boundary conditions can be written thus:

$$-\lambda_q(\nabla t)_w + q_q(\tau) - (1-\varepsilon)\,\varrho\, q_m(\tau) = 0\,, \tag{6.41}$$

$$\lambda_m(\nabla \theta)_w + \lambda_m \delta_1(\nabla t)_w + q_m(\tau) = 0\,, \tag{6.42}$$

where the subscript w denotes the surface of the body.

The first term of equality (6.41) corresponds to the quantity of heat which passes from the surface inwards to the body, the second term $q_q(\tau)$ corresponds to the quantity of heat transferred to the surface of the body, and the last term to the quantity of heat expended in the evaporation of the flux of liquid, equal to $(1-\varepsilon)\,q_m(\tau)$. If evaporation proceeds only inside the body ($\varepsilon = 1$), i.e. if it is vapour which is transferred to the surface, then this term will be absent. If, on the other hand, the flux of matter consists of a flux of liquid ($\varepsilon = 0$), then evaporation proceeds only on the surface. Equality (6.42) expresses the balance of mass of matter. From the surface of the body into the surrounding media there is a mass flux $q_m(\tau)$, and to the surface of the body matter is communicated under the influence of the potential gradient θ (first term of the equality) and under the action of the temperature gradient (second term of the equality).

In the case where the body interacts with a gaseous medium, the exchange of heat and mass proceeds according to the law of convection; then the fluxes $q_q(\tau)$ and $q_m(\tau)$ can be replaced by the following expressions

$$q_q(\tau) = \alpha_q(t_c - t_w)\,, \tag{6.43}$$

$$q_m(\tau) = \alpha_m(\theta_w - \theta_p) = \beta\gamma_0(u_w - u_p)\,, \tag{6.44}$$

where t_c and θ_p are respectively the temperature of the medium and its equilibrium potential of mass exchange, α_q and α_m are respectively the coefficients of heat exchange (kcal/m². hr. deg) and mass exchange (kg/m². hr. unit of potential), and β is the coefficient of mass exchange in relation to the difference of mass content (m/hr), equal to

$$\beta = \frac{\alpha_m}{c_m\,\gamma_0}\,. \tag{6.45}$$

The coefficient β is analogous to the coefficient of heat exchange α_h, in relation to the difference of the enthalpies, $\alpha_h = \alpha_q/c_q\,\gamma_0$.†

If we substitute the relations (6.43) and (6.44) in the corresponding conditions (6.41) and (6.42), then they will be called boundary conditions of the third kind. If the coefficients of heat and mass exchange are very great ($\alpha_q \to \infty$ and $\alpha_m \to \infty$), or more exactly, when the criteria Bi_q and Bi_m tend to infinity, then instead of conditions of the third kind we obtain boundary conditions of the first kind (t_w and θ_w are given as functions of the time).

The boundary conditions written in the form (6.41) and (6.42), where $q_q(\tau)$ and $q_m(\tau)$ are arbitrary functions of the time, are called boundary conditions of the second kind. In the particular case the fluxes of heat and matter can be constant ($q_q = \text{const.}$, $q_m = \text{const.}$).

At the initial moment of time ($\tau = 0$) the distribution of the potentials of heat and mass transfer are usually given

† In the following chapters for convenience the subscripts $q, 0$, will be omitted.

as functions of the coordinates, $t(x, y, z, 0)$ and $\theta(x, y, z, 0)$. In particular cases the distribution of potentials is assumed to be uniform ($t = \text{const.}$, $\theta = \text{const.}$) or in unidimensional problems according to a parabolic law, which corresponds to a first degree of approximation to the more complicated initial distribution of the potentials as functions of the coordinate.

§ 7. Generalized System of Differential Equations

The system of differential equations of heat and mass transfer deduced in § 6 can be written in the same criterial form as was done in § 4. Let us consider this in more detail.

The differential equations of heat and mass transfer describe the mechanism of the redistribution of heat and matter in the body. They represent the most general relation between the physical quantities characterizing the phenomena of transfer (the aggregate of phenomena described by a single transfer mechanism). Hence, we do not find in the differential equations any information about the concrete values of individual quantities which characterize any particular phenomenon. The variables which enter into the equations can take most varied values, each of which corresponds to some single physical situation. Hence, the system of differential equations is a mathematical model of a whole class of phenomena. On integration we obtain an indefinite number of different solutions which satisfy the given system of equations. In order to obtain from the large number of solutions a single particular solution which corresponds to a definite concrete phenomenon, it is necessary to have additional data, not contained in the initial system of equations. For this purpose it is necessary to know all the concrete peculiarities of a particular phenomenon which distinguish it from the whole class of similar phenomena. These supplementary conditions, which, in conjunction with the differential equations, uniquely determine the phenomenon, are called the conditions of uniqueness.

The initial and boundary conditions are also included amon the conditions of uniqueness and also the dimensions an form of the body and its physical characteristics and thos of the surrounding medium.

A large number of variables enter into the differentia equations of transfer and the conditions of uniqueness, sinc the phenomena of heat and mass transfer are very complicated Not all of these variables, however, are essential to the develop ment of the process of heat and mass transfer. For ever problem we study, including the problems of transfer, ther exists a collection of variables characteristic of the problem all of which must be taken into consideration. If we expres the problem in terms of these magnitudes we can reduce th number of quantities essential to the process to a small i.e. the smallest possible, number of arguments and functions The theory of similarity has as its object the working ou of general methods of determining such variables.

The basic idea from which these methods are developed consists of the replacement of the real process by the simples fictitious process, distinguished by the fact that for it al the differential operators remain constant in value both i space and time. In this case the expressions for the derivative can be replaced by relationships of finite form. For example suppose we have $y = f(x)$. Suppose that when x change from 0 to l and correspondingly y changes from 0 to L th condition $dy/dx = \text{const.}$ is satisfied all the time. Obviousl in this case $dy/dx = L/l$. If the condition $d^2y/dx^2 = \text{const}$ were observed, then, obviously, we should have $d^2y/dx^2 \sim L/l^2$ where $\sim$ is the sign of proportionality.

Let us illustrate this idea from the elementary exampl of the heat exchange of a body in the form of a plate wit a surrounding gasiform medium. We have

$$\alpha(t_w - t_c) = -\lambda \left(\frac{\partial t}{\partial x}\right)_w . \qquad (7.1$$

The coefficient of heat conductivity λ of the plate is suppose

constant. Let us put $\partial t/\partial x = \text{const}$. This condition is not in the present case a fictitous one, it exists for a stationary temperature field. Then we can write:

$$\frac{dt}{dx} = \frac{\Delta_l t}{l}, \tag{7.2}$$

where $\Delta_l t$ is the fall in temperature through the thickness of the plate l.

Let us denote the fall of temperature $(t_w - t_c)$ by Δt and call it the head of temperature. Then we can write:

$$d\Delta t = \lambda \frac{\Delta_l t}{l}; \quad \frac{\Delta_l t}{\Delta t} = \frac{dl}{\lambda}. \tag{7.3}$$

In the general case with a curvilinear non-stationary distribution of temperature this simple result ceases to apply, since now it is impossible to represent the derivative $\partial t/\partial x$ in the form $\Delta_l t/l$. However, if we put

$$\left(\frac{\partial t}{\partial x}\right)_w = -\epsilon \frac{\Delta_l t}{l}, \tag{7.4}$$

then the multiplier ϵ (the degree of distortion) will depend on the configuration of the curve of temperature distribution. Hence for all mutually similar temperature distributions ϵ has the same value. Consequently, the expression (7.3) can be rewritten in the form:

$$\frac{\Delta_l t}{\Delta t} = \frac{1}{\epsilon} \frac{\alpha l}{\lambda}, \tag{7.5}$$

and, for determining $\Delta_l t/l$ in the general case, it is necessary to know ϵ, which for all similar distributions is one and the same. Hence if we have found the value of ϵ for any single distribution, then it applies also to all distributions to which it is similar. A simple consideration shows that those distributions are similar to which the same value of $\alpha l/\lambda$ corresponds. This means that the coefficient ϵ is a single-valued function of the quantity $\alpha l/\lambda$. Consequently, equation (7.5) can be represented in the form

$$\frac{\Delta_l t}{\Delta t} = f\left(\frac{\alpha l}{\lambda}\right). \tag{7.6}$$

It is necessary to note that we have here represented the solution in the form of a function of a single argument, although, obviously, the distribution of temperature is determined by the influence of three parameters α, l and λ. The real significance of this fact is, that corresponding to our representation of the physical nature of the process, it is not each of these parameters alone which possesses the significant value but their common $\alpha l/\lambda$.

The expression $\alpha l/\lambda$ is a characteristic example of the specific variable of the type in which we are interested. In the theory of similarity it is usual to call such variables criteria of similarity and to denote them by the first two letters of the surname of a scientific man who has particularly contributed to the development of this department of knowledge. The expression of $\alpha l/\lambda$ is usually called the criterion of Biot in honour of the distinguished scientist of that name (Bi $= \alpha l/\lambda$).

The relation (7.6) can be rewritten thus

$$\frac{\Delta_l t}{\Delta t} = f(\mathrm{Bi})\,. \tag{7.7}$$

The transition from the original variables α, l, λ to the new variable Bi has as its consequence not only a diminution in the number of arguments. It results also in a radical change in the very setting-out of the analysis itself. Actually, if we work with the quantities α, l, λ, for each combination of given values of these parameters we obtain a certain particular case. In contrast with this a given value of Bi does not at all correspond to any particular case, since this value can be realized through an infinite number of different combinations of the quantities α, l, λ. Hence, by giving a particular value to Bi we determine not one concrete phenomenon, but an infinite number of different phenomena. Thus, in the new conception, the individual case corresponding to a given value of the argument is not a single phenomenon but an infinitely large group of phenomena all similar to one another. In this sense we can say that the new (specific) variables are generalized, and the whole analysis acquires a generalized character.

Let us return to our problem. The non-stationary process of heat exchange between the plate and the surrounding medium is not determined by the value of the single criterion Bi. The processes in different bodies will not develop synchronously since the tempo of change of the temperature field depends on λ, l, c. Consequently, the problem arises of the correct determination of the corresponding moments of time. For this purpose we must consider the equation of heat conductivity. For a body in the form of a plate it has the form

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} . \qquad (7.8)$$

Let us apply our method of transition to a fictitous scheme with constant values of the derivatives. The derivative $\partial t/\partial \tau$ is replaced by $\Delta_\tau t/\tau$, and the derivative $\partial^2 t/\partial x^2$ by $\Delta_l t/l^2$, where *the subscripts* τ *and* l *indicate the temperature change in* time τ and in distance l. Consequently, for the scheme we are considering we can write:

$$\frac{\Delta_\tau t}{\Delta_l t} = \frac{a\tau}{l^2} . \qquad (7.9)$$

If, at any instant, the temperature distributions are similar, then this similarity is preserved only on the condition that the ratios between the space changes and the time changes always remain the same. Hence it follows that the expression $a\tau/l^2$ must have the same value for all the bodies.

With a given initial temperature distribution any subsequent distribution depends on the duration of the process τ, the coefficient of temperature conductivity a, and the dimensions l. The essential value is not that of each of these quantities alone, but their definite combination $a\tau/l^2$. Obviously, the expression $a\tau/l^2$ is a specific generalized variable, i.e. is a criterion of similarity. It is usual to call this variable the Fourier criterion:

$$\mathrm{Fo} = \frac{a\tau}{l^2} . \qquad (7.10)$$

It has the significance of a generalized time. Hence it may be called the homochronous criterion (uniformity in time). If, for two bodies, the ratio l^2/a has one and the same value, then they become not only homochronous but synchronous. Then we can write:

$$\frac{\Delta_1 t}{\Delta t} = \varphi(\mathrm{Bi}, \mathrm{Fo}) . \tag{7.11}$$

This solution corresponds to the surface of the plate. For any point in the plate with coordinate x we can write the following expression for the temperature t at that point:

$$\frac{t}{t_0} = F\left(\mathrm{Bi}, \mathrm{Fo}, \frac{x}{l}\right), \tag{7.12}$$

where t_0 is the initial temperature.

The ratio x/l is called a criterion of parametric type or, simply, a parametric criterion. Obviously, the ratio t/t_0 is also a parametric criterion.

From the analysis of the above example we can draw the following important conclusions.

The solutions must be presented in the form of ratios connecting together the specific generalized variables, i.e. the criteria of similarity. These variables are either an immediate transcription of the condition of the problem (parametric criteria), or are found by a simple treatment of the fundamental equations. In all cases they are relative quantities: for parametric criteria we have the ratios of similar quantities, and in the remaining cases the criteria are a measure of the ratio of two physical effects of the same kind. Hence the criterial ratios have this important property, that they give the connection in non-dimensional form, and consequently their form does not depend on the choice of unit of measurement. The arguments of the criterial equations are criteria consisting of magnitudes determined by the conditions of the problem (determinative criteria). Criteria involving unknown magnitudes to be determined (non-determinative criteria) appear as functions.

It is shown in the theory of similarity that any non-determinative criterion is a single valued function of a combination of determinative criteria.

Solutions in the criterial form describe an infinite number of phenomena which, along with the initial conditions, are united into a single group when the following requirements are fulfilled: (1) geometrical similarity of the bodies; (2) similarity of their physical structure; (3) similarity of the initial conditions; (4) similarity of the conditions on the surface of interaction of the body with the surrounding medium.

We have obtained these conclusions on the basis of the analysis of a very elementary unidimensional problem of heat conduction. Nevertheless, they are completely general and completely rigorous.

Notice in particular that, by the constitution of criteria, complicated differential operators, corresponding to many-dimensional distributions, are replaced by a single component. Let us turn to the analysis of the system of differential equations of heat and mass transfer. From the analysis of the system of transfer equations (4.28) and (4.33) in binary gaseous mixtures we obtain the criteria of similarity of Soret (So) and Dufour (Du). The physical significance of these criteria was explained in § 4. From the system of differential equations (6.30) and (6.30′) of heat and mass transfer in capillary-porous bodies, we can obtain the following criteria of similarity:

1. Criterion of the lag of the temperature field relative to the transfer potential field of matter:

$$\mathrm{Lu} = \frac{a_m}{a_q}\,. \tag{7.13}$$

By means of this criterion the homochronous criterion of the potential field of transfer of matter (the mass exchange criterion of Fourier Fo_m) is connected with the homochronous criterion for the temperature field:

$$\mathrm{Fo}_m = \frac{a_m \tau}{l^2} = \mathrm{Fo}\,\mathrm{Lu}\,. \tag{7.14}$$

2. The Kossovich criterion, which is a specific form of the phase-change criterion and determines the relations between the heat expended in evaporation and the heat necessary for the heating of a moist body:

$$\mathrm{Ko} = \frac{\varrho \Delta u}{c_q \Delta T} = \frac{\varrho c_m \Delta \theta}{c_q \Delta T} . \tag{7.15}$$

3. The Posnov criterion Pn, which expresses the relative drop of mass-content and of the transfer potential of matter in a body, due to the drop of temperature:

$$\mathrm{Pn} = \frac{\delta_j \Delta \theta}{\Delta T} . \tag{7.16}$$

The Posnov criterion was first introduced as the relative drop in mass content of a body due to the drop of temperature:

$$\mathrm{Pn} = \frac{\delta \Delta u}{\Delta T} . \tag{7.17}$$

From the boundary conditions (6.41) and (6.42) along with (6.43) and (6.44) we obtain the two Kirpichev criteria: the heat exchange criterion Ki_q and the mass exchange criterion Ki_m, equal to

$$\mathrm{Ki}_q = \frac{q(\tau)\, l}{\lambda \Delta T} = \mathrm{Bi} \frac{t_a - t_w}{\Delta t}; \quad \mathrm{Ki}_m = \frac{g_m(\tau)\, l}{\lambda_m \Delta \theta} = \mathrm{Bi}_m \frac{\theta_w - \theta_a}{\Delta \theta}, \tag{7.18}$$

where $\mathrm{Bi}_m = \alpha_m l/\lambda_m = \beta l/\alpha_m$ is the mass exchange criterion of Biot. The quantities ΔT, Δu, $\Delta \theta$ are chosen to correspond with the conditions of the concrete problem. Criteria of similarity arising from the system of differential equations of heat and mass transfer (6.38), corresponding to the high intensity heating of moist capillary-porous bodies, are introduced in Chapter 6.

The system of differential equations of heat and mass transfer can be written in criterial form. In the case of transfer in a stationary, binary gas mixture, when the total deriva-

tive is equal to the partial derivative (for $\omega = 0$, $d/d\tau = \partial/\partial\tau$), equations (4.28) and (4.33) will take the form

$$\frac{\partial t^*}{\partial \mathrm{Fo}} = (1 + \mathrm{Lu}_D \mathrm{So}\, \mathrm{Du})\, \nabla^{*2} t^* + \mathrm{Lu}_D \mathrm{Du}\, \nabla^{*2} p_{10}^*, \quad (7.19)$$

$$\frac{\partial p_{10}^*}{\partial \mathrm{Fo}} = \mathrm{Lu}_D \mathrm{So}\, \nabla^{*2} t^* + \mathrm{Lu}_D \nabla^{*2} p_{10}^*, \quad (7.19')$$

where $$\mathrm{Lu}_D = D_{12}/a_q$$

The differential equations (6.30) and (6.30′) can be written in the criterial form as follows:

$$\frac{\partial t^*}{\partial \mathrm{Fo}} = \nabla^{*2} t^* + \varepsilon \mathrm{Ko} \frac{\partial \theta^*}{\partial \mathrm{Fo}}, \quad (7.20)$$

$$\frac{\partial \theta^*}{\partial \mathrm{Fo}} = \mathrm{Lu}_m(\nabla^{*2} \theta^* + \mathrm{Pn}\, \nabla^{*2} t^*), \quad (7.20')$$

where $\mathrm{Lu}_m = a_m/a_q$. The system of differential equations (6.38) in criterial form is introduced in Chapter 6.

All these systems of transfer equations can be written in the form of a single generalized system:

$$\frac{\partial \Pi_1^*}{\partial \mathrm{Fo}} = K_{11} \nabla^{*2} \Pi_1 + K_{12} \nabla^{*2} \Pi_2^* + K_{13} \nabla^{*2} \Pi_3^*, \quad (7.21)$$

$$\frac{\partial \Pi_2^*}{\partial \mathrm{Fo}} = K_{21} \nabla^{*2} \Pi_1 + K_{22} \nabla^{*2} \Pi_2 + K_{23} \nabla^{*2} \Pi_3, \quad (7.21')$$

$$\frac{\partial \Pi_3^*}{\partial \mathrm{Fo}} = K_{31} \nabla^{*2} \Pi_1 + K_{32} \nabla^{*2} \Pi_2^2 + K_{33} \nabla^{*2} \Pi_3, \quad (7.21'')$$

where Π_i^* is the corresponding relative transfer potential, and K_{ij} are criterial coefficients.

For the first system $\Pi_1^* = t^*$, $\Pi_2 = p_{10}^*$, $\Pi_3^* = 0$, $K_{11} = 1 + \mathrm{Du}\,\mathrm{So}\,\mathrm{Lu}_D$, $K_{12} = \mathrm{Du}\,\mathrm{Lu}_D$, $K_{21} = \mathrm{So}\,\mathrm{Lu}_D$, $K_{22} = \mathrm{Lu}_D$. For the system of equations of high-intensity transfer in a capillary-porous body, $\Pi_1^* = t^*$, $\Pi_2^* = \theta^*$, $\Pi_3^* = p^*$, and the corresponding coefficients are equal to

$$K_{11} = 1 + \varepsilon \mathrm{Ko}\, \mathrm{Lu}_m \mathrm{Pn}; \quad K_{12} = \varepsilon \mathrm{Ko}\, \mathrm{Lu}_m; \quad K_{13} = \varepsilon \mathrm{Bu}\, \mathrm{Lu}_p;$$

$$K_{21} = \mathrm{Lu}_m \mathrm{Pn}; \quad K_{22} = \mathrm{Lu}_m; \quad K_{23} = \mathrm{Lu}_p \frac{\mathrm{Bu}}{\mathrm{Ko}};$$

$$K_{31} = -\frac{1}{\mathrm{Bu}}\varepsilon\,\mathrm{Ko}\,\mathrm{Lu}_m\,\mathrm{Pn};$$

$$K_{32} = -\frac{1}{\mathrm{Bu}}\varepsilon\,\mathrm{Ko}\,\mathrm{Lu}_m;\ K_{33} = (1-\varepsilon)\,\mathrm{Lu}_p. \qquad (7.22)$$

The system of equations (4.20) is obtained from the system of equations (4.21) if, in the latter, we put Π_3^* equal to 0.

The system of differential equations (4.21) was obtained by considering the processes of heat and mass transfer in their mutual relation by the methods of the thermodynamics of irreversible processes. These methods enable us to consider, in their mutual relations, not only the transfer of heat and mass, but also the transfer of electric charges, of the kinetic energy of the macroscopic movement of gases or liquids (phenomenon of internal friction) and a number of other processes of molecular transfer. In this case it is to be expected that we shall find a more general system of differential equations of transfer of the type

$$\begin{aligned}
\frac{\partial \Pi_1^*}{\partial \mathrm{Fo}} &= \sum_{i=1}^{n} K_{1i}\,\nabla^{*2}\,\Pi_i^*,\\
\frac{\partial \Pi_2^*}{\partial \mathrm{Fo}} &= \sum_{i=1}^{n} K_{2i}\,\nabla^{*2}\,\Pi_i^*,\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
\frac{\partial \Pi_n^*}{\partial \mathrm{Fo}} &= \sum_{i=1}^{n} K_{ni}\,\nabla^{*2}\,\Pi_i^*.
\end{aligned} \qquad (7.23)$$

The working out of methods of solving such a system of differential equations as (7.23) is a topical problem in the creation of a mathematical theory of the thermodynamics of irreversible processes. Hence, the solutions set out in the following chapters for particular cases are of general interest for the further development of transfer phenomena on the basis of the thermodynamics of irreversible processes.

CHAPTER 2

Non-stationary Fields of Transfer Potential of Heat and Mass with Boundary Conditions of the Third Kind

INTERNAL heat and mass exchange can be described by a system of partial differential equations of transfer (see Chapter 1) which, for one-dimensional bodies, can be written as follows:

$$\frac{\partial t(\xi,\tau)}{\partial\tau} = a_q\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial t(\xi,\tau)}{\partial\xi}\right] + \frac{\varepsilon\varrho\, c'_T}{c_q}\frac{\partial\vartheta(\xi,\tau)}{\partial\tau}, \qquad (I)$$

$$\frac{\partial\vartheta(\xi,\tau)}{\partial\tau} = a_m\left[\frac{\partial^2\vartheta(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial\vartheta(\xi,\tau)}{\partial\xi}\right] +$$

$$+ a_m\delta\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial t(\xi,\tau)}{\partial\xi}\right], \qquad (II)$$

where t is the temperature of the body (the transfer potential of heat),

ϑ is the transfer potential of mass,

c_q and c'_T are respectively the specific heat capacity and mass capacity,

$a_q = \frac{\lambda_q}{c\gamma_0}$ is the coefficient of temperature conductivity (the coefficient of conductivity of the transfer potential of heat),

$a_m = \frac{\lambda_m}{c'_T\gamma_0}$ is the coefficient of conductivity of the transfer potential of mass,

λ_m is the coefficient of mass conductivity,

δ is the Soret coefficient,

ϱ is the specific heat of phase change,

Γ is a form constant (for an infinite plate $\Gamma = 0$, for a cylinder $\Gamma = 1$, for a sphere $\Gamma = 2$),

ξ is a generalized coordinate chosen according to the shape of the body (for infinite plate and rod $\xi = x$, for an infinite cylinder and a sphere ε is the criterion of phase change $\xi = r$),

The system of differential equations (I′–II′) can be written in another form if, instead of the transfer potential of mass θ, we introduce the specific mass-content (moisture content) of the body, u:

$$\frac{\partial t(\xi, \tau)}{\partial \tau} = a_q \left[\frac{\partial^2 t(\xi, \tau)}{\partial \xi^2} + \frac{\Gamma}{\xi} \frac{\partial t(\xi, \tau)}{\partial \xi}\right] + \frac{\varepsilon \varrho}{c_q} \frac{\partial u(\xi, \tau)}{\partial \tau}, \qquad (\text{I}')$$

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = a_m \left[\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} + \frac{\Gamma}{\xi} \frac{\partial u(\xi, \tau)}{\partial \xi}\right] +$$

$$+ a_m \delta \left[\frac{\partial^2 t(\xi, \tau)}{\partial \xi^2} + \frac{\Gamma}{\xi} \frac{\partial t(\xi, \tau)}{\partial \xi}\right]. \qquad (\text{II}')$$

The system of equations (I–II) differs from the system (I′ — II′) only by the fact that in the first case δ denotes the Soret coefficient and in the second the thermal-gradient coefficient δ, and instead of the ratio of the mass-content to the heat-content c'_T/c_q in the last term of the first equation we have a magnitude which is the reciprocal of the heat-content.

Hence, solutions obtained for the system (I–II) can be used for the system (I′–II′) by changing the ratio c'_T/c_q into the quantity $1/c_q$, and changing θ into the specific moisture content u. Hence, for convenience of exposition we will consider the system (I–II), whilst in order to emphasize the close connection between the systems (I′–II′) and (I–II) we shall denote the ratio c'_T/c_q by $1/c$. Then, changing θ into u and c into c_q, we shall obtain the solutions of equations (I′–II′).

Analytical solutions of (I–II) for boundary conditions of the first kind and constant integral specific moisture content have been given by S. P. Kuznetsov and E. I. Sizyakova. Solutions for boundary conditions of the second kind in the absence of thermo-gradient mass transfer ($\delta = 0$) and arbitrary initial conditions have been found by M. S. Smirnov,

and solutions (I–II) for thermal-gradient boundary conditions of the second kind when the thermal-gradient coefficient differs from zero, found by A. P. Prudnikov. The complete system of equations for internal heat and mass transfer for boundary conditions of the third kind and a different law of temperature change of the medium with time were solved by Yu. A. Mikhailov, and for an infinite plate with $t_a = \text{const}$. by A. V. Luikov. Special cases of the solution of the incomplete system of equations (I–II) have been given by F. M. Polonskaya, S. P. Kuznetsov, A. V. Luikov, P. D. Lebedev, Yu. A. Mikhailov and M. S. Smirnov.

The influence of individual criteria of thermal and mass similarity on the process of heat and mass transfer has been investigated by A. V. Luikov, Yu. A. Mikhailov, P. D. Lebedev and E. I. Sizyakova. S. P. Kuznetsov, E. I. Sizyakova and others, on the basis of the analytical solutions obtained, have worked out rapid methods for the experimental determination of the coefficients of heat and mass transfer. A characteristic feature of the abovenamed solutions is that they are all obtained by methods of integral transformation. Analysis and practical calculation show that classical methods do not permit of the effective separation of the variables t and θ and the pursuit of the process of solution by this means to the end. The use of methods of integral transformation successfully overcomes all these difficulties.

We give below solutions of the system of equations of heat and mass transfer for the classical one-dimensional bodies.

The boundary conditions of the system of differential equations (I–II) for the case of boundary conditions of the third kind can be written thus:

$$-\lambda \frac{\partial t(R,\tau)}{\partial \xi} + \alpha [t_c - t(R,\tau)] - (1-\varepsilon)\varrho\beta [\theta(R,\tau) - \theta_p] = 0, \quad \text{(III)}$$

$$a_m \frac{\partial \vartheta(R,\tau)}{\partial \xi} + a_m \delta \frac{\partial t(R,\tau)}{\partial \xi} + \beta [\theta(R,\tau) - \theta_p] = 0, \quad \text{(IV)}$$

where α is the coefficient of heat exchange,

β is the coefficient of mass exchange,

$$\frac{\partial t(0,\tau)}{\partial \xi} = \frac{\partial \theta(0,\tau)}{\partial \xi} = 0\,,$$

$$t\,(0,\tau) < \infty, \qquad \text{(V)}$$

$$\theta(0,\tau) < \infty,$$

$$t(\xi, 0) = f(\xi)\,; \quad \theta(\xi, 0) = \varphi(\xi)\,. \qquad \text{(VI)}$$

Solutions of the system of equations (I–II) under boundary conditions (III–VI) with constant and parabolic initial distribution of transfer potentials of heat and mass i.e.

$$f(\xi) = t_0 = \text{const.}, \quad \varphi(\xi) = \theta_0 = \text{const}\,,$$

$$f(\xi) = t_u + \frac{\xi^2}{R^2}(t_n - t_u), \quad \varphi(\xi) = \theta_u - \frac{\xi^2}{R^2}(\theta_u - \theta_n)\,,$$

are of particular significance.

For example, a constant initial distribution of transfer potentials can be regarded as the analogue of the warming of a moist body, and a parabolic initial distribution as the analogue of the period of decreasing rate of drying, etc.

§ 1. Non-stationary Fields of Transfer Potential of Heat and Mass in a Medium with Constant Temperature

We shall demonstrate the general method of solving the complete system of equations of internal heat and mass transfer by finding the field of transfer potential of heat and mass for bodies in the form of an infinite plate and a sphere. The solutions given in this chapter are obtained by means of Laplace transformations.

(a) *Infinite plate. Constant initial conditions*

($\Gamma = 0$, $\xi = x$, $f(x) = t_0 = \text{const.}$, $\varphi(x) = \theta_0 = \text{const.}$). Applying the Laplace transformation to equations (I–II) we obtain

$$s t_L(x, s) - t_0 = a_q\, t''_L(x, s) + \frac{\varepsilon\varrho}{c}\, s\, \theta_L(x, s) - \frac{\varepsilon\varrho}{c}\, \theta_0\,, \qquad (1.1)$$

$$s\, \theta_L(x, s) - \theta_0 = a_m\, \theta''_L(x, s) + a_m\, \delta\, t''_L(x, s)\,. \qquad (1.2)$$

We find the differential equation for the temperature $t_L(x, s)$ in the following way: from equation (1.1) we determine $\theta_L(x, s)$, its second derivative $\theta''_L(x, s)$, and substitute the expressions obtained in equation (1.2). After simplifying we obtain

$$t_L^{IV}(x, s) - \left(\frac{s}{a_q} + \frac{\varepsilon\varrho\delta}{a_q c} + \frac{s}{a_m}\right) t''_L(x, s) + \\ + \frac{s^2}{a_q a_m} t_L(x, s) - \frac{s t_0}{a_q a_m} = 0 . \qquad (1.3)$$

The general integral of the non-homogeneous ordinary differential equation (1.3) can be determined by once more using the Laplace transformation with respect to the coordinate x. The required operations give us

$$t_L(x, s) = \frac{t_0}{s} + C_1 e^{\sqrt{\left(\frac{s}{a_q}\right) r_1 x}} + C_2 e^{\sqrt{\left(\frac{s}{a_q}\right) r_1 x}} + \\ + C_3 e^{-\sqrt{\left(\frac{s}{a_q}\right) r_1 x}} + C_4 e^{-\sqrt{\left(\frac{s}{a_q}\right) r_1 x}}, \qquad (1.4)$$

where $C_k (k = 1, 2, 3, 4)$ are arbitrary constants and

$$r_j^2 = \frac{1}{2}\left\{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + \right. \\ \left. + (-1)^j \sqrt{\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}\right]}\right\}, \ (j = 1,2) . \quad (1.5)$$

$\mathrm{Fo} = \varepsilon \mathrm{Ko}\, \mathrm{Pn} = \frac{\varepsilon\varrho\delta}{c_q}$ Fedorov criterion;

$\mathrm{Ko} = \frac{c'_L \varrho(\theta_0 - \theta_p)}{c_q(t_a - t_0)}$ Kossovich criterion;

$\mathrm{Pn} = \frac{\delta(t_a - t_0)}{\theta_0 - \theta_p}$ Posnov criterion;

$\mathrm{Lu} = \frac{a_m}{a_q}$ criterion of inertia of fields of heat and mass potential.

The conditions of symmetry (V) in the transforms may be written:

$$t'_L(0, s) = \theta'_L(0, s) = 0 .$$

It follows from these conditions that $C_1 = C_3$ and $C_2 = C_4$.

That is, the solution for the transform of the temperature may be rewritten

$$t_L(x, s) = \frac{t_0}{s} + B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x + B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x . \quad (1.6)$$

Here B_1 and B_2 are new constants with respect to x.

A general solution for the transform of the mass transfer potential $\theta_L(x, s)$ we obtain by substituting $t''_L(x, s)$, found from equation (1.6), and $t_L(x, s)$ in (1.1). We have

$$\theta_L(x, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon\varrho} B_1(1 - v_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x +$$

$$+ \frac{c}{\varepsilon\varrho} B_2(1 - v_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x . \quad (1.7)$$

We determine B_1 and B_2 from the boundary conditions (III–IV), which become, after the application of the Laplace transformation:

$$- \lambda t'_L(R, s) + \alpha \left[\frac{tc}{s} - t_L(R, s)\right] -$$

$$- (1 - \varepsilon) \varrho\beta \left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0 , \quad (1.8)$$

$$\lambda_m \theta'_L(R, s) + \lambda_m \delta t'_L(R, s) + \beta \left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0 . \quad (1.9)$$

From equations (1.6) and (1.7) we find $t_L(R, s)$, $t'_L(R, s)$, $\theta_L(R, s)$, $\theta'_L(R, s)$ and substitute the values obtained in the boundary conditions (1.8) and (1.9). As a result we have:

$$- B_1 \left\{\lambda \sqrt{\left(\frac{s}{a_q}\right)} v_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R +\right.$$

$$\left. + \left[\alpha + \frac{1 - \varepsilon}{\varepsilon} c\beta(1 - v_1^2)\right] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R \right\} -$$

$$- B_2 \left\{\lambda \sqrt{\left(\frac{s}{a_q}\right)} v_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R +\right.$$

$$\left. + \left[\alpha + \frac{1 - \varepsilon}{\varepsilon} c\beta(1 - v_2^2)\right] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R \right\} +$$

$$+ \frac{\alpha(tc - t_0)}{s} - (1 - \varepsilon) \varrho\beta \frac{\theta_0 - \theta_p}{\varepsilon} = 0 , \quad (1.10)$$

$$B_1 \left\{ \left[\lambda_m \frac{c}{\varepsilon\varrho} (1 - v_1^2) + \lambda_m \delta \right] \sqrt{\left(\frac{s}{a_q}\right)} v_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R + \right.$$

$$\left. + \beta \frac{c}{\varepsilon\varrho} (1 - v_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R \right\} +$$

$$+ B_2 \left\{ \left[\lambda_m \frac{c}{\varepsilon\varrho} (1 - v_2^2) + \lambda_m \delta \right] \sqrt{\left(\frac{s}{a_q}\right)} v_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R + \right.$$

$$\left. + \beta \frac{c}{\varepsilon\varrho} (1 - v_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R \right\} + \beta \frac{\theta_0 - \theta_p}{s} = 0 . \qquad (1.11)$$

After simplification of the general elements of (1.10) and (1.11) and their treatment by the criteria of similarity to determine B_1 and B_2, we obtain the system of algebraic equations (1.12)

$$\left.\begin{aligned} B_1 Q_1 + B_2 Q_2 - \frac{tc - t_0}{s} + (1 - \varepsilon) \frac{\varrho\beta}{\alpha} \frac{\theta_0 - \theta_p}{s} = 0 \\ B_1 P_1 + B_2 P_2 + \frac{\mathrm{Fe}}{\delta} \frac{\theta_0 - \theta_p}{s} = 0 \end{aligned}\right\}, \qquad (1.12)$$

where

$$Q_j = [1 + (1 - v_j^2) K_1] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_j R +$$

$$+ \frac{1}{\mathrm{Bi}_q} \sqrt{\left(\frac{s}{a_q}\right)} v_j R \sinh \sqrt{\left(\frac{s}{a_p}\right)} v_j R , \qquad (1.13)$$

$$P_j = (1 - v_j^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_j R +$$

$$+ [(1 - v_j^2) + \mathrm{Fe}] \frac{1}{\mathrm{Bi}_m} \sqrt{\left(\frac{s}{a_q}\right)} v_j R \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_j R . \qquad (1.14)$$

$\mathrm{Bi}_q = \frac{\alpha}{\lambda_q} R$ and $\mathrm{Bi}_m = \frac{\beta}{\lambda_m} R$ are the heat and mass criteria of Biot,

$k_1 = \frac{1 - \varepsilon}{\varepsilon} \mathrm{Lu} \frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}$ is a complex criterion.

Solution of the system of equations (1.12) gives

$$B_1 = - \frac{(tc - t_0) P_2 + \frac{\varepsilon\varrho}{c} (\theta_0 - \theta_p) (Q_2 - K_1 P_2)}{s(Q_2 P_1 - Q_1 P_2)},$$

$$B_2 = \frac{(tc - t_0) P_1 + \frac{\varepsilon\varrho}{c} (\theta_0 - \theta_p) (Q_1 - K_1 P_1)}{s(Q_2 P_1 - Q_1 P_2)} . \qquad (1.15)$$

Consequently

$$t_L(x, s) - \frac{t_0}{s} = (t_c - t_0)\left\{\frac{P_1 + \varepsilon \mathrm{Ko}\,(Q_1 - K_1 P_1)}{s(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x - \right.$$

$$\left. - \frac{P_2 + \varepsilon \mathrm{Ko}\,(Q_2 - K_1 P_2)}{s(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x\right\}, \qquad (1.16)$$

$$\frac{\theta_0}{s} - \theta_L(x, s) = (\theta_0 - \theta_p)\left\{\frac{Q_2 + \frac{1}{\varepsilon \mathrm{Ko}} P_2 - K_1 P_2}{s(Q_2 P_1 - Q_1 P_2)} \times \right.$$

$$\times (1 - \nu_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x - \frac{Q_1 + \frac{1}{\varepsilon \mathrm{Ko}} P_1 - K_1 P_1}{s(Q_2 P_1 - Q_1 P_2)} \times$$

$$\left. \times (1 - \nu_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x\right\}. \qquad (1.17)$$

We have obtained the general transformed solution of the system of differential equations of heat and mass transfer (I–VI). To obtain the inversion of the solution we can use the generalized theorem of expansion. The theorem of expansion is valid for transforms which can be represented in the form of the ratio of two generalized polynomials $\Phi_0(s)/\psi_0(s)$ and also for the ratio of two non-generalized polynomials $\Phi_1(s)$ and $\psi_1(s)$ when these, by multiplication or division by s^l, ($|l| \leqslant 1$), can be reduced to the former. Here it is supposed that the function $\psi_0(s)$ has only simple roots, and that the degree of the polynomial in the denominator is one greater than that of the polynomial in the numerator.

In our case each of the transformed equations can be expressed in the form of the ratio of two polynomials

$$\frac{1}{t_c - t_0}\left[t_L(x, s) - \frac{t_0}{s}\right] = \frac{\Phi_{01}(s)}{\psi_0(s)}, \qquad (1.18)$$

$$\frac{1}{\theta_0 - \theta_p}\left[\frac{\theta_0}{s} - \theta_L(x, s)\right] = \frac{\Phi_{02}(s)}{\psi_0(s)}. \qquad (1.19)$$

The denominator $\psi_0(s) = s(Q_2 P_1 - Q_1 P_2)$ in the equations (1.18) and (1.19) is the same, and the polynomials in the nume-

rators are respectively equal to:

$$\Psi_{01}(s) = [P_1 + \varepsilon \mathrm{Ko}\,(Q_1 - K_1 P_1)] \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 x -$$

$$- [P_2 + \varepsilon \mathrm{Ko}\,(Q_2 - K_1 P_2)] \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 x\,,$$

$$\Psi_{02}(s) = \left(Q_2 + \frac{P_2}{\varepsilon \mathrm{Ko}} - K_1 P_2\right)(1 - r_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 x -$$

$$- \left(Q_1 + \frac{P_1}{\varepsilon \mathrm{Ko}} - K_1 P_1\right)(1 - r_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 x\,.$$

The numerators and denominators in equations (1.18) and (1.19) are generalized polynomials with respect to s; hence, as analysis shows, the transition from the transform to the inversion can be made by the ordinary expansion theorem, i.e.

$$L^{-1}\left[\frac{\Psi_{01}(s)}{\psi_0(s)}\right] = \frac{\Psi_{01}(0)}{\psi_0'(0)} + \sum_{n=1}^{\infty} \frac{\Psi_{01}(s_n)}{\psi_0'(s_n)} \exp(s_n \tau) \qquad (1.18')$$

and

$$L^{-1}\left[\frac{\Psi_{02}(s)}{\psi_0(s)}\right] = \frac{\Psi_{02}(0)}{\psi_0'(0)} + \sum_{n=1}^{\infty} \frac{\varphi_{02}(s_n)}{\psi_0'(s_n)} \exp(s_n \tau)\,. \qquad (1.19')$$

Here the symbol L^{-1} indicates the inverse Laplace transformation.

Let us find the roots of $\psi_0(s)$, for which purpose we put

$$\psi_0(s) = s(Q_2 P_1 - Q_1 P_2) = 0\,.$$

We obtain:

(1) $s = s_0 = 0$ (zero root)

(2) $s = s_n$ (where s_n satisfies the equation $Q_2 P_1 - Q_1 P_2 = 0$).

Let us make the values of the roots s_n more precise. For this purpose we will replace the hyperbolic sines and cosines in the expressions (1.13) and (1.14) according to the formulae $\sinh z = -i \sin iz$ and $\cosh z = \cos iz$. We obtain

$$s_n = -\frac{a_q \mu_n^2}{R^2}\,,$$

where μ_n are the roots of the characteristic equation,

$$Q_{n2} P_{n1} - Q_{n1} P_{n2} = 0 , \tag{1.20}$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n , \tag{1.21}$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - [(1 - \nu_j^2) + \mathrm{Fc}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n . \tag{1.22}$$

To obtain the inversions of the transforms (1.18′) and (1.19′) we shall determine the quantities $\psi_0'(0)$, $\psi_0'(s_n)$, $\Phi_{01}(0)$, $\Phi_{02}(0)$, $\Phi_{01}(s_n)$, $\Phi_{02}(s_n)$ which they contain. Calculation gives

$$\psi_0' = \nu_2^2 - \nu_1^2 ,$$

$$\Phi_{01}(0) = \Phi_{02}(0) = \nu_2^2 - \nu_1^2 ,$$

$$\Phi_{01}(s_n) = - \left\{ [P_{n2} + \varepsilon \, \mathrm{Ko} \, (Q_{n2} - K_1 P_{n2})] \cos \nu_1 \mu_n \frac{x}{R} - \right.$$

$$\left. - [P_{n1} + \varepsilon \, \mathrm{Ko} \, (Q_{n1} - K_1 P_{n1}) \cos \nu_2 \mu_n \frac{x}{R} \right\} ,$$

$$\Phi_{02}(s_n) = - \left\{ \left(Q_{n1} + \frac{1}{\varepsilon \, \mathrm{Ko}} P_{n1} - K_1 P_{n1} \right) \times \right.$$

$$\times (1 - \nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} - \left(Q_{n2} + \frac{1}{\varepsilon \, \mathrm{Ko}} P_{n2} - K_1 P_{n2} \right) \times$$

$$\left. \times (1 - \nu_1^2) \cos \nu_1 \mu_n \frac{x}{R} \right\} ,$$

$$\psi_0'(s_n) = \frac{1}{2} \mu_n \psi_n ,$$

where

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2} , \tag{1.23}$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1 \right] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n , \tag{1.24}$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n + \frac{(1 - \nu_j^2) + \mathrm{Fc}}{\mathrm{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n) . \tag{1.25}$$

Let us substitute the values found above in equations (1.18′) and (1.19′); as a result, for non-stationary fields of

transfer potential of heat and mass, we obtain

$$\frac{t(x,\tau)-t_0}{t_c-t_0}=1-2\sum_{n=1}^{\infty}\left\{\frac{P_{n2}+\varepsilon\,\mathrm{Ko}\,(Q_{n2}-K_1P_{n2})}{\mu_n\varphi_n}\times\right.$$

$$\times\cos\nu_1\mu_n\frac{x}{R}-\frac{P_{n1}+\varepsilon\,\mathrm{Ko}\,(Q_{n1}-K_1P_{n1})}{\mu_n\varphi_n}\times$$

$$\left.\times\cos\nu_2\mu_n\frac{x}{R}\right\}\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (1.26)$$

$$\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=1-2\sum_{n=1}^{\infty}\left\{\frac{Q_{n1}+\frac{1}{\varepsilon\,\mathrm{Ko}}P_{n1}-K_1P_{n1}}{\mu_n\varphi_n}\times\right.$$

$$\times(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}-\frac{Q_{n2}+\frac{1}{\varepsilon\,\mathrm{Ko}}P_{n2}-K_1P_{n2}}{\mu_n\varphi_n}\times$$

$$\left.\times(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right\}\exp\left(-\mu_n^2\,\mathrm{Fo}\right). \qquad (1.27)$$

Thus, the system of differential equations for the heat and mass transfer (I–VI) for an infinite plate under boundary conditions of the third kind with constant initial distribution of transfer potentials throughout the material have been completely solved. For convenience in use we rewrite the solution, collecting together equations (1.5) and (1.20) to (1.27):

$$T=\frac{t(x,\tau)-t_0}{t_c-t_0}=1-$$

$$-\sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R}-C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \quad (1.28)$$

$$\Theta=\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=1-\sum_{n=1}^{\infty}\left[C_{n1}^{*}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}-\right.$$

$$\left.-C_{n2}^{*}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right]\exp\left(-\mu_n^2\,\mathrm{Fo}\right). \qquad (1.29)$$

Here ν_j $(j=1;\,2)$ are coefficients determined by the relation

$$\nu_j^2=\frac{1}{2}\left\{\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)+\right.$$

$$\left.+(-1)^j\sqrt{\left[\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)^2-\frac{4}{\mathrm{Lu}}\right]}\right\},$$

C_{nj} and C^*_{nj} are determined by the relations:

$$C_{nj} = 2\frac{P_{nj} + \varepsilon \mathrm{Ko}\,(Q_{nj} - K_1 P_{nj})}{\mu_n \psi_n}, \qquad C^*_{nj} = \frac{C_{nj}}{\varepsilon \mathrm{Ko}}, \tag{1.30}$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n + \\ + \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n),$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n,$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - [(1 - \nu_j^2) + \mathrm{Fe}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n.$$

μ_n are the roots of the characteristic equation

$$Q_{n2} P_{n1} = Q_{n1} P_{n2}. \tag{1.20}$$

The values of the characteristic roots μ_n can be obtained by a graphic-analytical method. For this purpose we will rewrite equation (1.20) in a somewhat different form:

$$\frac{M}{N} = \frac{\mu}{\mathrm{Bi}_q}, \tag{1.31}$$

$$M = P_1 \cos \nu_2 \mu - P_2 \cos \nu_1 \mu,$$

where

$$N = \left[(1 - \nu_1^2) \frac{1 - \varepsilon}{\varepsilon} \mathrm{Lu}\, \mathrm{Bi}_m \frac{\cos \nu_1 \mu}{\mu} - \nu_1 \sin \nu_1 \mu\right] P_2 - \\ - \left[(1 - \nu_2^2) \frac{1 - \varepsilon}{\varepsilon} \mathrm{Lu}\, \mathrm{Bi}_m \frac{\cos \nu_2 \mu}{\mu} - \nu_2 \sin \nu_2 \mu\right] P_1.$$

If the left-hand side of the expression (1.31) is denoted by $\Phi_\mu = \Phi(\mu)$ and the right-hand side by $(y = 1/\mathrm{Bi}_q)\ \mu$, then the points of intersection of the curves Φ_μ with the line y in the graph of $[\mu, \Phi_\mu]$ will give the values of the roots μ_n for the given combination of the criteria Lu, ε, Ko, Pn, Bi_m, Bi_q (Fig. 1)

For different values of the criterion Bi_q (from 0·10 to ∞) for various values of Lu, Bi_m, Fe and ε, the characteristic roots μ_n are collected in Tables 1–4. For convenience in

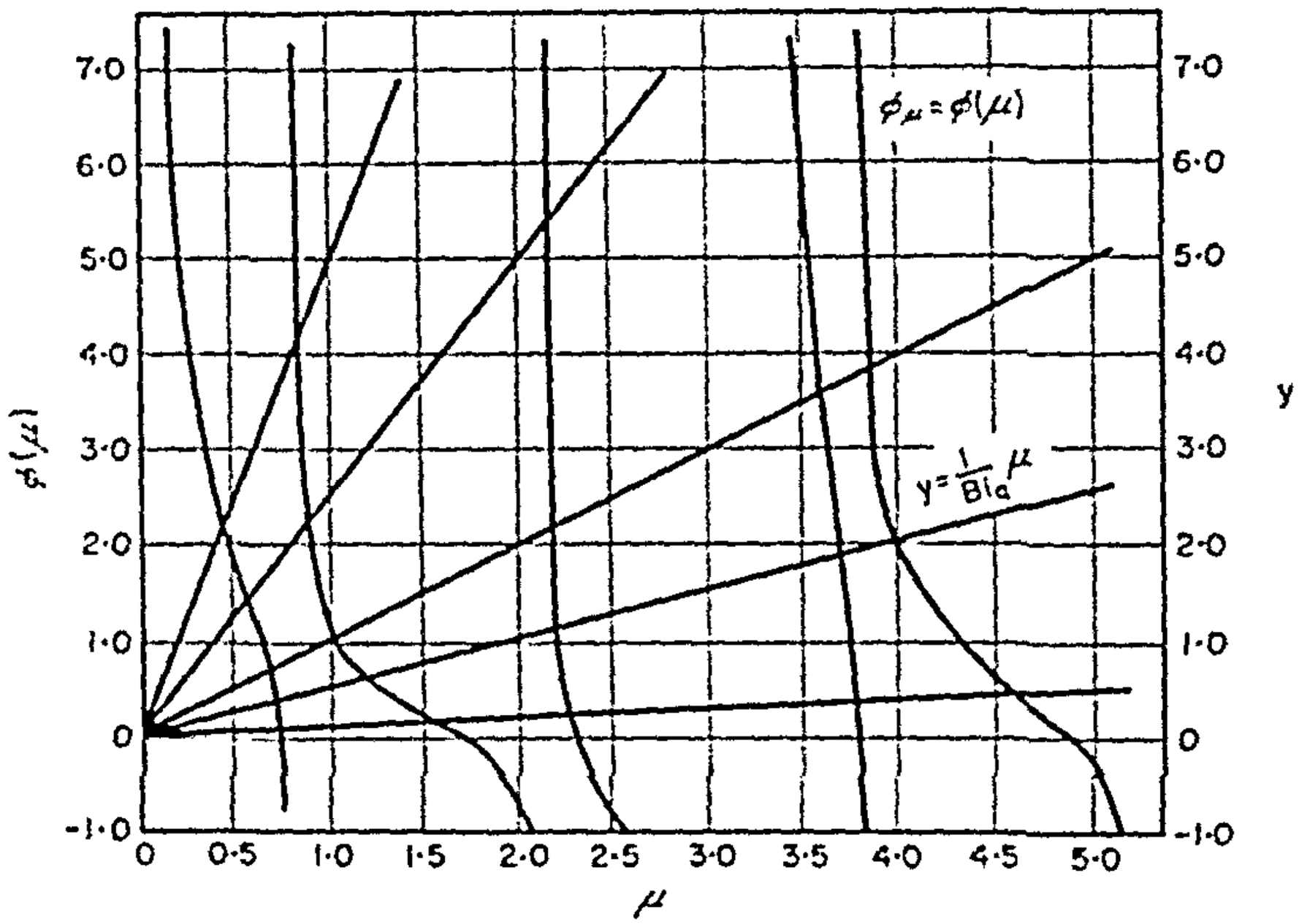


FIG. 1. Method of determining the roots of the characteristic equation $\Phi(\mu) = \dfrac{1}{\mathrm{Bi}_q}\mu$ for an infinite plate

practical use graphs of the dependence of μ_n on the various indicated criteria are given in Figs. 2–9.

From consideration of equation (1.31) and Fig. 1 it follows that we have an infinite number of roots μ_n of increasing value:

$$\mu_1 < \mu_2 < \ldots \mu_n < \ldots .$$

In virtue of this inequality, the infinite series in equations (1.28) and (1.29) converge fairly rapidly. As Table 5 and Fig. 10 show, with the increase in the non-dimensional time the percentage error introduced by neglecting the terms of the series which are under the summation sign in comparison with the two first terms diminishes.

TABLE 1. ROOTS OF THE CHARACTERISTIC EQUATION $\Phi(\mu) = (1/\mathrm{Bi}_q)\,\mu$ FOR VARIOUS VALUES OF Lu (INFINITE PLATE), $\mathrm{Fo} = \varepsilon\,\mathrm{KoPn} = 0{\cdot}3$, $\varepsilon = 0{\cdot}5$, $\mathrm{Bi}_m = 10$.

Bi_q	Lu = 0·15		Lu = 0·3					Lu = 0·5			Lu = 0·7		Lu = 1·0		Lu = 2·0	
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_4	μ_5	μ_1	μ_2	μ_3	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·10	0·306	0·569	0·307	0·814				0·307			0·307		0·308		0·308	
0·15	0·367	0·576	0·370	0·818	2·168			0·371	1·077	2·508	0·372	1·307	0·373		0·373	
0·20	0·415	0·583	0·423	0·825	2·169	3·365	3·809	0·424	1·084	2·511	0·425	1·314	0·426	1·637	0·424	
0·25	0·451	0·596	0·465	0·832	2·170	3·379	3·810	0·467	1·090	2·515	0·468	1·318	0·469	1·643	0·468	
0·30	0·477	0·613	0·500	0·839	2·171	3·396	3·811	0·505	1·096	2·519	0·506	1·322	0·507	1·649	0·507	
0·35	0·494	0·635	0·530	0·847	2·172	3·408	3·812	0·537	1·101	2·523	0·540	1·329	0·542	1·655	0·543	
0·40	0·506	0·658	0·556	0·856	2·173	3·423	3·814	0·568	1·104	2·527	0·571	1·335	0·573	1·660	0·574	
0·45	0·513	0·682	0·579	0·867	2·174	3·434	3·816	0·593	1·107	2·534	0·596	1·342	0·599	1·666	0·602	
0·50	0·518	0·706	0·599	0·877	2·175	3·444	3·818	0·616	1·112	2·537	0·620	1·347	0·623	1·674	0·627	
0·55	0·522	0·730	0·616	0·888	2·176	3·454	3·820	0·637	1·120	2·542	0·642	1·358	0·646	1·677	0·649	
0·60	0·525	0·752	0·630	0·898	2·178	3·468	3·822	0·654	1·128	2·546	0·661	1·359	0·666	1·681	0·669	
0·70	0·529	0·796	0·652	0·925	2·180	3·491	3·825	0·688	1·147	2·552	0·696	1·372	0·702	1·689	0·709	
0·80	0·532	0·834	0·668	0·950	2·183	3·515	3·830	0·716	1·167	2·557	0·727	1·384	0·734	1·701	0·745	
0·90	0·533	0·869	0·681	0·973	2·187	3·537	3·836	0·738	1·183	2·564	0·753	1·396	0·762	1·710	0·774	
1·0	0·535	0·903	0·690	0·999	2·191	3·558	3·841	0·757	1·198	2·572	0·775	1·407	0·786	1·717	0·799	
1·5	0·538	1·028	0·716	1·106	2·195	3·639	3·878	0·817	1·269	2·600	0·857	1·459	0·878	1·753	0·900	

TABLE 1. *Contd.*

Bi_q	Lu = 0·15		Lu = 0·3					Lu = 0·5			Lu = 0·7		Lu = 1·0		Lu = 2·0	
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_4	μ_5	μ_1	μ_2	μ_3	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2·0	0·539	1·114	0·726	1·188	2·197	3·685	3·936	0·848	1·329	2·625	0·901	1·503	0·937	1·780	0·967	
3·0	0·539	1·238	0·735	1·298	2·204	3·725	4·074	0·877	1·412	2·657	0·951	1·566	0·996	1·817	1·047	
4·0	0·540	1·311	0·739	1·364	2·213	3·742	4·187	0·891	1·469	2·682	0·974	1·606	1·033	1·841	1·091	2·760
5·0	0·540	1·362	0·741	1·409	2·220	3·749	4·281	0·900	1·509	2·698	0·989	1·637	1·057	1·858	1·125	2·660
6·0	0·540	1·398	0·742	1·440	2·224	3·753	4·357	0·906	1·541	2·711	0·999	1·659	1·072	1·871	1·148	2·623
7·0	0·541	1·425	0·743	1·465	2·229	3·756	4·420	0·907	1·561	2·720	1·007	1·675	1·083	1·879	1·163	2·595
8·0	0·541	1·444	0·743	1·487	2·231	3·759	4·479	0·912	1·570	2·728	1·011	1·689	1·092	1·887	1·178	2·576
9·0	0·541	1·459	0·744	1·504	2·233	3·760	4·518	0·914	1,595	2·735	1·015	1·700	1·099	1·893	1·188	2·561
10	0·541	1·472	0·744	1·516	2·235	3·762	4·554	0·915	1·606	2·740	1·017	1·709	1·106	1·898	1·197	2·552
15	0·542	1·509	0·745	1·552	2·241	3·764	4·665	0·919	1·638	2·756	1·027	1·734	1·122	1·912	1·224	2·526
20	0·542	1·530	0·746	1·572	2·245	3·766	4·730	0·922	1·653	2·766	1·033	1·748	1·128	1·919	1·237	2·512
30	0·542	1·550	0·747	1·597	2·248	3·767	4·780	0·924	1·675	2·775	1·036	1·764	1·138	1·928	1·253	2·498
40	0·542	1·558	0·747	1·607	2·251	3·768	4·826	0·925	1·684	2·782	1·039	1·774	1·143	1·933	1·261	2·490
50	0·542	1·565	0·747	1·614	2·253	3·769	4·840	0·926	1·691	2·786	1·040	1·780	1·146	1·936	1·266	2·485
∞	0·542	1·580	0·748	1·638	2·256	3·770	4·911	0·929	1·711	2·801	1·045	1·707	1·155	1·943	1·284	2·469

TABLE 2. ROOTS OF THE CHARACTERISTIC EQUATION $\Phi(\mu) = (1/\mathrm{Bi}_q)\,\mu$ FOR VARIOUS VALUES OF Bi_m (INFINITE PLATE), Lu = 0·3, KoPn = 0·6, $\varepsilon = 0{\cdot}5$.*

Bi_q	$\mathrm{Bi}_m = 0{\cdot}1$		$\mathrm{Bi}_m = 0{\cdot}5$		$\mathrm{Bi}_m = 5$		$\mathrm{Bi}_m = 7$		$\mathrm{Bi}_m = 15$		$\mathrm{Bi}_m = 20$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·10	0·313	1·640	0·302	0·370	0·307	0·746	0·307	0·783	0·307	0·839	0·307	0·854
0·15	0·379	1·641	0·339	0·401	0·370	0·752	0·370	0·786	0·370	0·844	0·370	0·859
0·20	0·434	1·642	0·348	0·449	0·419	0·759	0·422	0·793	0·423	0·850	0·423	0·865
0·25	0·482	1·643	0·350	0·493	0·460	0·767	0·463	0·800	0·465	0·856	0·466	0·870
0·30	0·524	1·644	0·352	0·534	0·497	0·774	0·499	0·809	0·501	0·863	0·502	0·876
0·35	0·562	1·646	0·352	0·571	0·526	0·783	0·529	0·816	0·531	0·871	0·533	0·884
0·40	0·595	1·647	0·352	0·606	0·550	0·793	0·554	0·826	0·557	0·878	0·559	0·892
0·45	0·626	1·648	0·353	0·637	0·570	0·805	0·577	0·838	0·581	0·888	0·583	0·902
0·50	0·653	1·649	0·353	0·664	0·586	0·818	0·595	0·850	0·601	0·898	0·603	0·912
0·55	0·680	1·650	0·353	0·691	0·602	0·832	0·610	0·863	0·618	0·908	0·622	0·923
0·60	0·705	1·652	0·353	0·716	0·615	0·847	0·623	0·875	0·633	0·918	0·637	0·934
0·70	0·751	1·655	0·354	0·762	0·632	0·877	0·643	0·900	0·657	0·941	0·661	0·955
0·80	0·790	1·657	0·354	0·803	0·643	0·906	0·658	0·927	0·676	0·965	0·680	0·975
0·90	0·826	1·659	0·354	0·839	0·651	0·937	0·668	0·954	0·690	0·990	0·694	0·998
1·0	0·859	1·660	0·354	0·872	0·657	0·965	0·676	0·981	0·701	1·014	0·706	1·020

TABLE 2. *Contd.*

Bi_q	$Bi_m = 0{\cdot}1$		$Bi_m = 0{\cdot}5$		$Bi_m = 5$		$Bi_m = 7$		$Bi_m = 15$		$Bi_m = 20$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1·5	0·984	1·671	0·354	0·997	0·672	1·031	0·695	1·094	0·730	1·117	0·737	1·121
2·0	1·070	1·682	0·355	1·088	0·679	1·166	0·704	1·117	0·742	1·197	0·750	1·200
3·0	1·180	1·701	0·355	1·198	0·684	1·278	0·711	1·285	0·752	1·302	0·760	1·308
4·0	1·245	1·718	0·355	1·260	0·686	1·348	0·714	1·358	0·756	1·373	0·765	1·379
5·0	1·286	1·730	0·355	1·304	0·687	1·393	0·716	1·403	0·759	1·420	0·768	1·425
6·0	1·316	1·740	0·355	1·337	0·687	1·426	0·717	1·436	0·760	1·450	0·770	1·455
7·0	1·336	1·751	0·355	1·359	0·688	1·451	0·718	1·462	0·761	1·476	0·771	1·481
8·0	1·352	1·761	0·355	1·377	0·688	1·473	0·718	1·482	0·762	1·499	0·772	1·504
9·0	1·363	1·767	0·355	1·388	0·689	1·487	0·719	1·496	0·762	1·513	0·773	1·519
10	1·370	1·773	0·355	1·396	0·689	1·499	0·720	1·508	0·763	1·526	0·774	1·531
15	1·397	1·795	0·355	1·428	0·690	1·537	0·721	1·546	0·765	1·564	0·775	1·570
20	1·409	1·807	0·355	1·444	0·690	1·557	0·721	1·566	0·765	1·585	0·776	1·589
30	1,421	1·822	0·355	1·458	0·690	1·579	0·722	1·586	0·766	1·606	0·777	1·611
40	1·427	1·829	0·355	1·465	0·690	1·589	0·722	1·596	0·766	1·616	0·777	1·621
50	1·430	1·835	0·355	1·470	0·690	1·595	0·722	1·603	0·767	1·622	0·778	1·627
∞	1·443	1·861	0·355	1·485	0·691	1·621	0·723	1·628	0·767	1·650	0·778	1·652

† For $Bi_m = 10$ see Table 1.

TABLE 3. ROOTS OF THE CHARACTERISTIC EQUATION $\Phi(\mu) = (1/Bi_q)\mu$ FOR VARIOUS VALUES OF $Fe = \varepsilon\, Ko\, Pn$ (INFINITE PLATE), $Lu = 0{\cdot}3$, $\varepsilon = 0{\cdot}5$, $Bi_m = 10$.†

Bi_q	$Fe = 0{\cdot}1$		$Fe = 0{\cdot}2$		$Fe = 0{\cdot}4$		$Fe = 0{\cdot}5$		$Fe = 0{\cdot}6$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·10	0·310	0·793	0·309	0·803	0·306	0·824	0·305	0·835	0·304	0·845
0·15	0·375	0·794	0·373	0·807	0·368	0·832	0·366	0·845	0·364	0·858
0·20	0·428	0·796	0·426	0·811	0·419	0·840	0·415	0·853	0·412	0·870
0·25	0·474	0·800	0·469	0·816	0·461	0·848	0·455	0·862	0·451	0·881
0·30	0·513	0·803	0·507	0·820	0·494	0·855	0·488	0·874	0·482	0·895
0·35	0·548	0·806	0·539	0·826	0·523	0·866	0·516	0·887	0·510	0·908
0·40	0·579	0·809	0·567	0·833	0·547	0·877	0·541	0·899	0·534	0·925
0·45	0·606	0·813	0·592	0·841	0·569	0·889	0·562	0·913	0·554	0·939
0·50	0·630	0·818	0·613	0·849	0·587	0·902	0·578	0·927	0·570	0·954
0·55	0·650	0·825	0·632	0·858	0·604	0·914	0·593	0·940	0·583	0·968
0·60	0·669	0·834	0·647	0·869	0·617	0·928	0·605	0·954	0·594	0·982
0·70	0·696	0·852	0·671	0·891	0·633	0·954	0·626	0·982	0·614	1·009
0·80	0·715	0·873	0·689	0·916	0·654	0·980	0·641	1·012	0·629	1·036
0·90	0·728	0·897	0·702	0·940	0·666	1·007	0·653	1·039	0·641	1·065
1·0	0·737	0·922	0·711	0·964	0·676	1·032	0·662	1·064	0·650	1·091
1·5	0·753	1·033	0·733	1·071	0·701	1·139	0·687	1·171	0·676	1·205

TABLE 3. *Contd.*

Bi_q	Fo = 0·1		Fo = 0·2		Fo = 0·4		Fo = 0·5		Fo = 0·6	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2·0	0·760	1·119	0·742	1·153	0·712	1·219	0·699	1·251	0·688	1·286
3·0	0·765	1·227	0·748	1·264	0·721	1·326	0·710	1·360	0·699	1·390
4·0	0·768	1·301	0·752	1·333	0·726	1·393	0·714	1·427	0·704	1·457
5·0	0·769	1·351	0·753	1·380	0·728	1·441	0·717	1·472	0·707	1·501
6·0	0·770	1·383	0·754	1·409	0·729	1·470	0·719	1·501	0·709	1·531
7·0	0·770	1·410	0·755	1·438	0·730	1·496	0·720	1·524	0·710	1·553
8·0	0·770	1·431	0·756	1·459	0·731	1·577	0·721	1·543	0·711	1·571
9·0	0·770	1·443	0·756	1·475	0·732	1·581	0·722	1·559	0·712	1·584
10	0·770	1·459	0·757	1·485	0·733	1·581	0·723	1·567	0·713	1·595
15	0·770	1·504	0·758	1·530	0·734	1·584	0·725	1·607	0·714	1·631
20	0·770	1·527	0·758	1·551	0·735	1·602	0·726	1·624	0·716	1·648
30	0·771	1·554	0·758	1·574	0·735	1·625	0·726	1·644	0·716	1·666
40	0·771	1·564	0·759	1·584	0·736	1·633	0·727	1·653	0·717	1·676
50	0·771	1·574	0·759	1·593	0·736	1·641	0·727	1·661	0·717	1·683
∞	0·771	1·603	0·759	1·618	0·737	1·665	0·728	1·683	0·718	1·704

† For Fo = 0·3 see Table 1.

TABLE 4. ROOTS OF THE CHARACTERISTIC EQUATION $\Phi(\mu) = (1/\mathrm{Bi}_q)\,\mu$ FOR VARIOUS VALUES OF ε (INFINITE PLATE), Lu = 0·3, Ko Pn = 0·6, $\mathrm{Bi}_m = 10^{\dagger}$

Bi_q	$\varepsilon = 0{\cdot}1667$		$\varepsilon = 0{\cdot}3333$		$\varepsilon = 0{\cdot}6667$		$\varepsilon = 0{\cdot}8333$		$\varepsilon = 1{\cdot}000$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·10	0·307	0·836	0·307	0·825	0·307	0·803	0·307	0·793	0·307	0·783
0·15	0·373	0·840	0·372	0·830	0·371	0·809	0·371	0·798	0·370	0·789
0·20	0·423	0·846	0·423	0·835	0·422	0·815	0·422	0·806	0·421	0·797
0·25	0·465	0·852	04·65	0·841	0·464	0·823	0·464	0·813	0·463	0·802
0·30	0·501	0·858	0·500	0·849	0·499	0·831	0·498	0·819	0·497	0·812
0·35	0·533	0·866	0·532	0·855	0·529	0·839	0·528	0·830	0·526	0·821
0·40	0·559	0·872	0·558	0·863	0·555	0·848	0·553	0·840	0·551	0·832
0·45	0·583	0·881	0·581	0·873	0·577	0·859	0·575	0·852	0·572	0·845
0·50	0·603	0·890	0·601	0·883	0·595	0·870	0·593	0·863	0·591	0·857
0·55	0·622	0·900	0·619	0·894	0·614	0·882	0·611	0·876	0·607	0·871
0·60	0·638	0·910	0·634	0·905	0·628	0·894	0·625	0·888	0·620	0·885
0·70	0·662	0·931	0·658	0·927	0·650	0·919	0·645	0·915	0·639	0·912
0·80	0·681	0·952	0·676	0·951	0·664	0·945	0·658	0·942	0·653	0·941
0·90	0·695	0·975	0·689	0·974	0·675	0·973	0·670	0·971	0·664	0·970
1·0	0·705	0·999	0·698	0·999	0·683	0·999	0·677	0·999	0·671	0·999
1·5	0·753	1·033	0·733	1·071	0·701	1·139	0·697	1·171	0·696	1·205

TABLE 4. *Contd.*

Bi_q	$\varepsilon = 0{\cdot}1667$		$\varepsilon = 0{\cdot}3333$		$\varepsilon = 0{\cdot}6667$		$\varepsilon = 0{\cdot}8333$		$\varepsilon = 1{\cdot}000$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2·0	0·745	1·172	0·733	1·170	0·716	1·190	0·707	1·197	0·700	1·203
3·0	0·755	1·273	0·743	1·285	0·726	1·304	0·715	1·314	0·707	1·322
4·0	0·759	1·340	0·747	1·360	0·728	1·379	0·718	1·387	0·710	1·400
5·0	0·761	1·382	0·749	1·393	0·730	1·425	0·720	1·436	0·712	1·450
6·0	0·762	1·414	0·751	1·427	0·732	1·459	0·721	1·475	0·713	1·487
7·0	0·763	1·435	0·752	1·450	0·733	1·484	0·722	1·499	0·714	1·511
8·0	0·764	1·454	0·753	1·470	0·734	1·505	0·723	1·520	0·715	1·534
9·0	0·765	1·467	0·754	1·486	0·734	1·521	0·723	1·537	0·715	1·552
10	0·766	1·477	0·754	1·499	0·735	1·536	0·724	1·553	0·716	1·566
15	0·767	1·513	0·756	1·532	0·736	1·572	0·725	1·590	0·717	1·606
20	0·767	1·533	0·757	1·553	0·736	1·593	0·725	1·612	0·717	1·629
30	0·768	1·554	0·757	1·573	0·737	1·615	0·726	1·634	0·717	1·653
40	0·768	1·564	0·758	1·583	0·737	1·626	0·726	1·645	0·718	1·664
50	0·769	1·575	0·758	1·593	0·738	1·637	0·727	1·657	0·718	1·676
∞	0·770	1·605	0·758	1·617	0·738	1·666	0·727	1·684	0·719	1·707

† For $\varepsilon = 0{\cdot}5$ see Table 1.

TABLE 5. FIRST LINE OF TABLE % ERROR

Fo	% error			
	1—T		1—Θ	
	$x/R = 0$	$x/R = 1$	$x/R = 0$	$x/R = 1$
0·1	11·30	40·16	18·05	137·37
0·2	6·62	20·53	18·72	71·46
0·3	4·20	12·74	9·23	40·05
0·4	3·21	8·48	6·00	23·48
0·5	2·26	5·75	3·85	14·15
0·6	1·59	3·91	2·47	8·68
0·7	1·10	2·65	1·58	5·37
0·8	0·76	1·79	1·01	3·35
0·9	0·52	1·20	0·64	2·10
1·0	0·35	0·80	0·41	1·32

Beginning from a definite value of the Fourier criterion, determined by the degree of accuracy decided upon in advance, it is permissible to use only the first two terms of the expansion. Thus, for the system of equations with constant initial conditions, assuming an error of 1·0 and 0·5 per cent, the initial values of Fo for simplified calculations are given in Table 6.

This simplification is of great importance, as it leads to a very great reduction in the volume of calculation to be performed, and renders it possible to present general solutions in a form convenient for practical application. The initial values of Fo for the simplified computation vary with changes in the combination of criteria involved in the solution.

As an analysis of the computations shows, for any given initial value of Fo, the greatest influence in simplifying the calculation is exerted by the criterion Lu. As Lu increases, the initial value of Fo required for simplification diminishes. The criteria ε, Ko and Pn exert a considerably weaker influence. As will be seen from Table 6, the initial value of Fo for local magnitudes increases, and for mean values diminishes,

TABLE 6. INITIAL VALUES OF Fo FOR SIMPLIFIED CALCULATION OF HEAT AND MASS TRANSFER.
(Lu = 0·3, Ko = 1·2, Pn = 0·5, ε = 0·5, Bi_m = 10)

First column Heading	Bi_q	Fo								Δ %
		$1-T$		$1-\Theta$		$1-\overline{T}$	$1-\overline{\Theta}$	$\frac{d\overline{T}}{d\,\mathrm{Fo}}$	$\frac{d\overline{\Theta}}{d\,\mathrm{Fo}}$	
		$x/R=0$	$x/R=1$	$x/R=0$	$x/R=1$					
First Sub-heading Plate, Constant Initial Conditions.	1	0·75	0·66	0·79	1·06	0·48	0·59	1·03	~ 1·13	1
		0·79	0·81	0·94	1·19	0·61	0·74	1·19	~ 1·31	0·5
	10	0·73	0·95	0·80	1·06	0·11	0·57	0·29	1·06	1
		0·89	1·08	0·95	1·19	0·17	0·73	0·37	1·19	0·5
Second Sub-heading Sphere, Constant Initial Conditions	10		0·24		0·27	0·17	0·13	0·26	0·22	1
			0·27		0·31	0·21	0·15	0·30		0·5
Third Sub-heading Plate, Parabolic Initial Conditions	10	0·66	0·87	0·73	1·00	0·13	0·50	0·25	0·33	1
		0·84	1·4	0·88	1·11	0·16	0·64	0·33	1·15	0·5

Last column Reading Error (∇) %

with increase of Bi_q. The influence of the Biot criterion is especially noticeable in the case of the non-dimensional heat potential. In an analogous way, the initial value of Fo for the non-dimensional mass potential depends to a particular

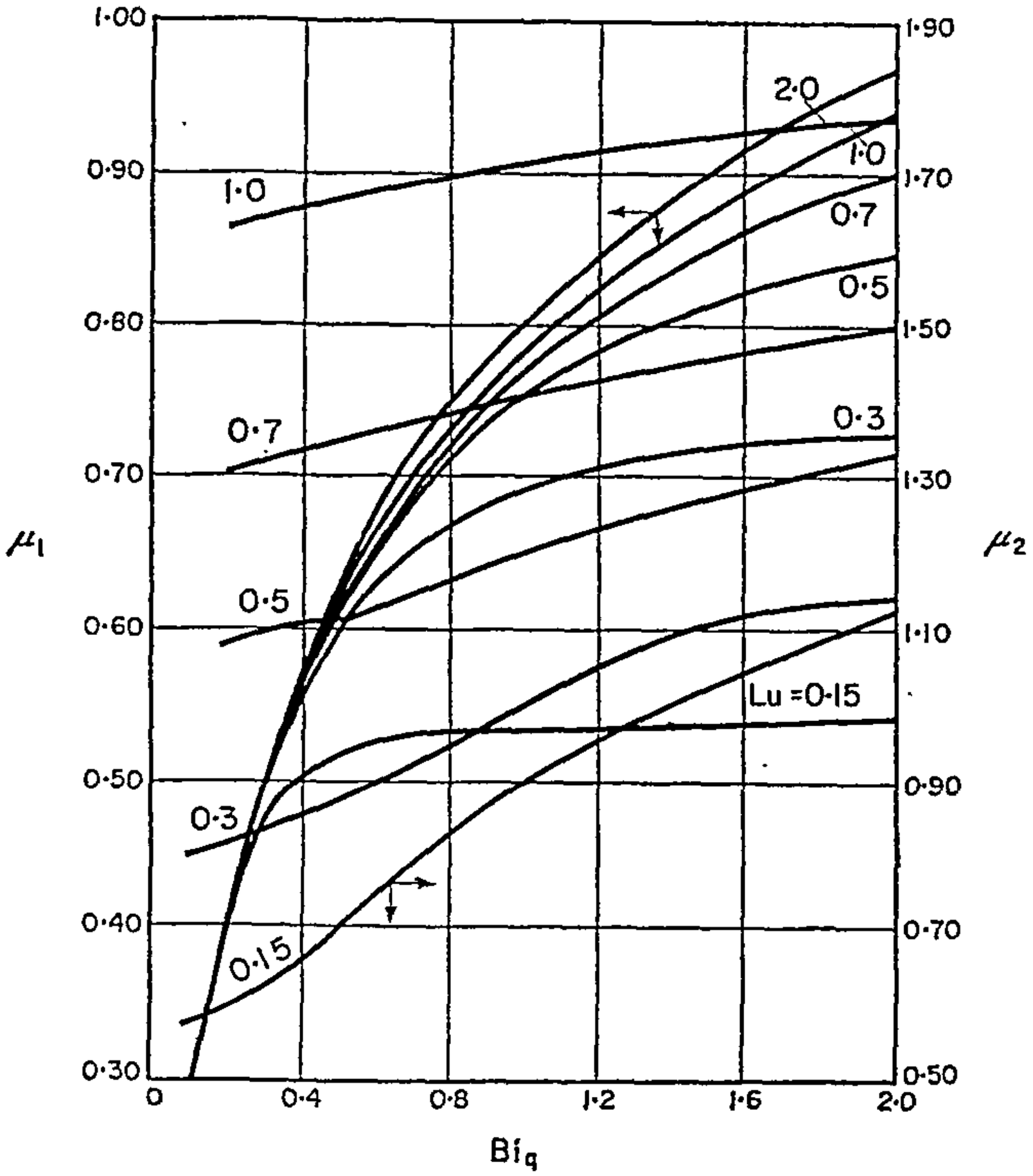


FIG. 2. Relation of the characteristic roots to Lu for an infinite plate ($Bi_q = 0{\cdot}1$ to $2{\cdot}0$)

degree on the criterion Bi_m. In a general way, the initial value of the criterion Fo for simplification is less for heat exchange than for mass exchange.

Thus, from what has been said it follows that the non-dimensional potentials of transfer in an infinite plate are functions of the relative coordinate, the non-dimensional time, and also the criteria Lu, Bi_q, Bi_m, ε, Ko, and Pn, i.e.

$$T = T(x/R, \text{Fo}, \text{Lu}, Bi_q, Bi_m, \varepsilon, \text{Ko}, \text{Pn}), \qquad (1.32)$$

$$\Theta = \Theta(x/R, \text{Fo}, \text{Lu}, Bi_q, Bi_m, \varepsilon, \text{Ko}, \text{Pn}), \qquad (1,33)$$

since

$$C_{nJ} = \psi_1(\mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Bi}_m, \varepsilon, \mathrm{Ko}, \mathrm{Pn}), \tag{1.34}$$

$$C^*_{nJ} = \psi_2(\mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Bi}_m, \varepsilon, \mathrm{Ko}, \mathrm{Pn}). \tag{1.35}$$

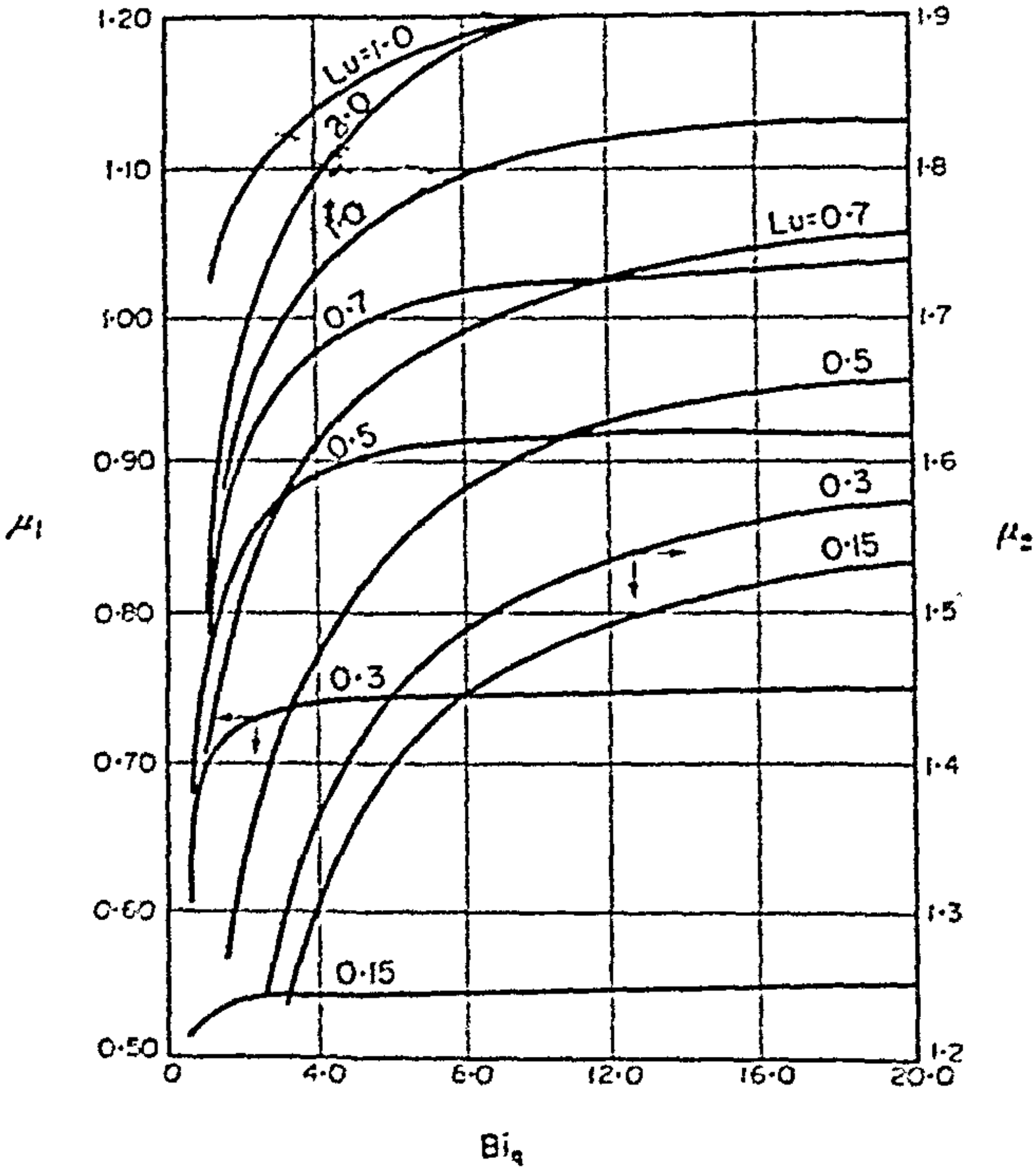


FIG. 3. Relation of the characteristic roots to Lu for an infinite plate ($\mathrm{Bi}_q = 1$ to 20)

The numerical values of the two first coefficients C_{nJ} and C^*_{nJ} as they depend on the criteria mentioned are given in Tables 7–12, and for ease of calculation in Figs. 11–16.

In studying the processes of internal heat and mass transfer it is necessary to know the mean temperature and transfer potential of the mass, and also the local and mean rates at which they change. For example, the mean temperature permits us to calculate by the relation

$$\Delta Q = c\gamma_0[\bar{t}(\tau) - t_0]$$

the expenditure of heat in raising the temperature of unit volume of the body. The connection between the mean and local rates of change of the transfer potentials enables us to obtain characteristics which are important for a number of

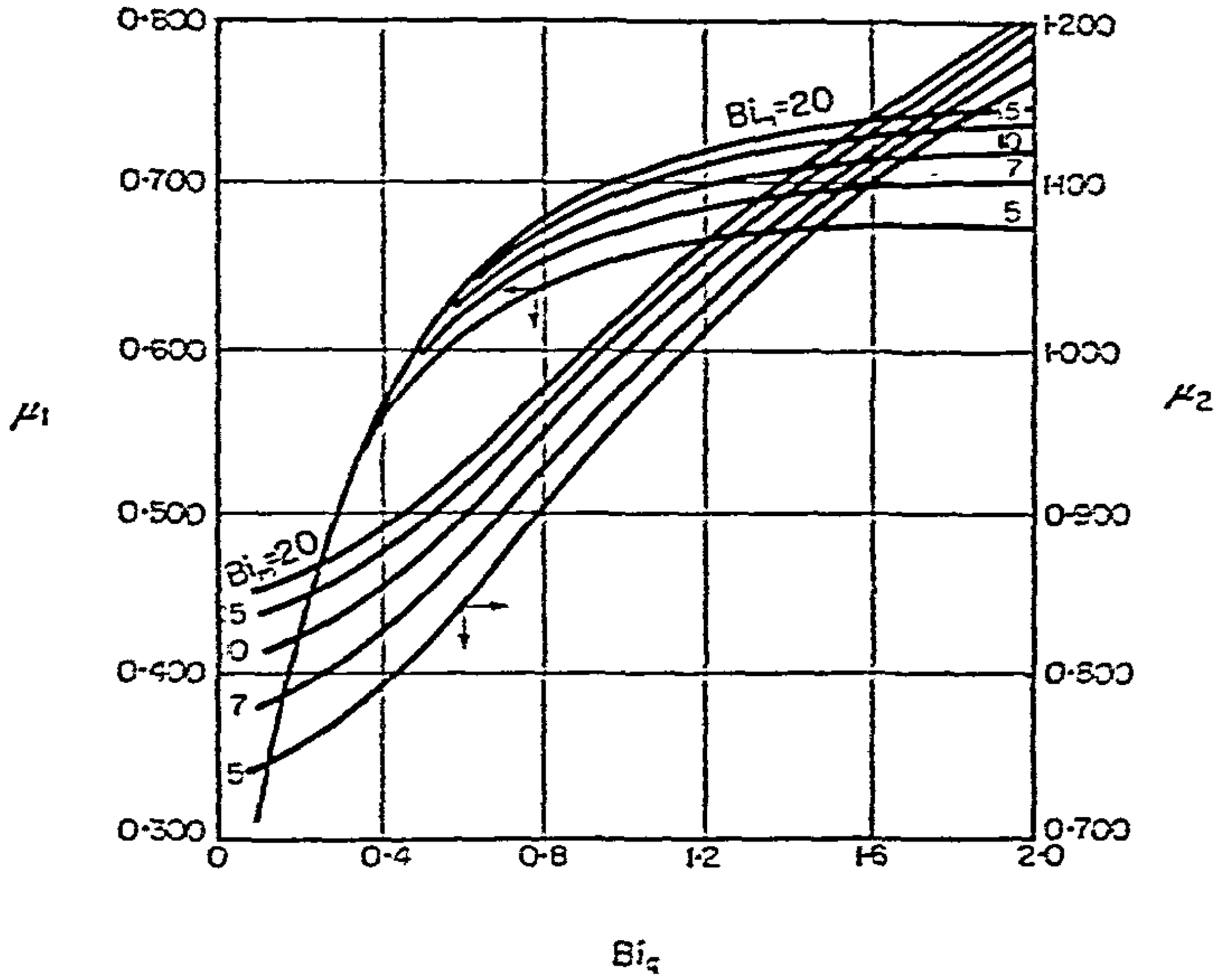


FIG. 4. Relation of the characteristic roots to Bi_m for an infinite plate ($Bi_q = 0{\cdot}2$ to $2{\cdot}0$)

processes (for drying, for example, the temperature curve and the curve of rate of drying).

The mean values of the transfer potentials in an infinite plate will be obtained from the relations

$$\bar{t}(\tau) = \frac{1}{R}\int_0^R t(x,\tau)\,dx,\ \bar{\Theta}(\tau) = \frac{1}{R}\int_0^R \Theta(x,\tau)\,dx\,.$$

If for $t(x, \tau)$ and $\Theta(x, \tau)$ we substitute the corresponding expressions from (1.28) and (1.29), then after integration we obtain the following expressions for the mean values of the non-dimensional temperature and mass potential:

$$\bar{T} = \frac{\bar{t}(\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (1.36)$$

$$\overline{\Theta} = \frac{\theta_0 - \bar{\theta}(\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.37)$$

where

$$D_n = D_{n2} - D_{n1}, \; D_n^* = D_{n1}^* - D_{n2}^*, \qquad (1.38)$$

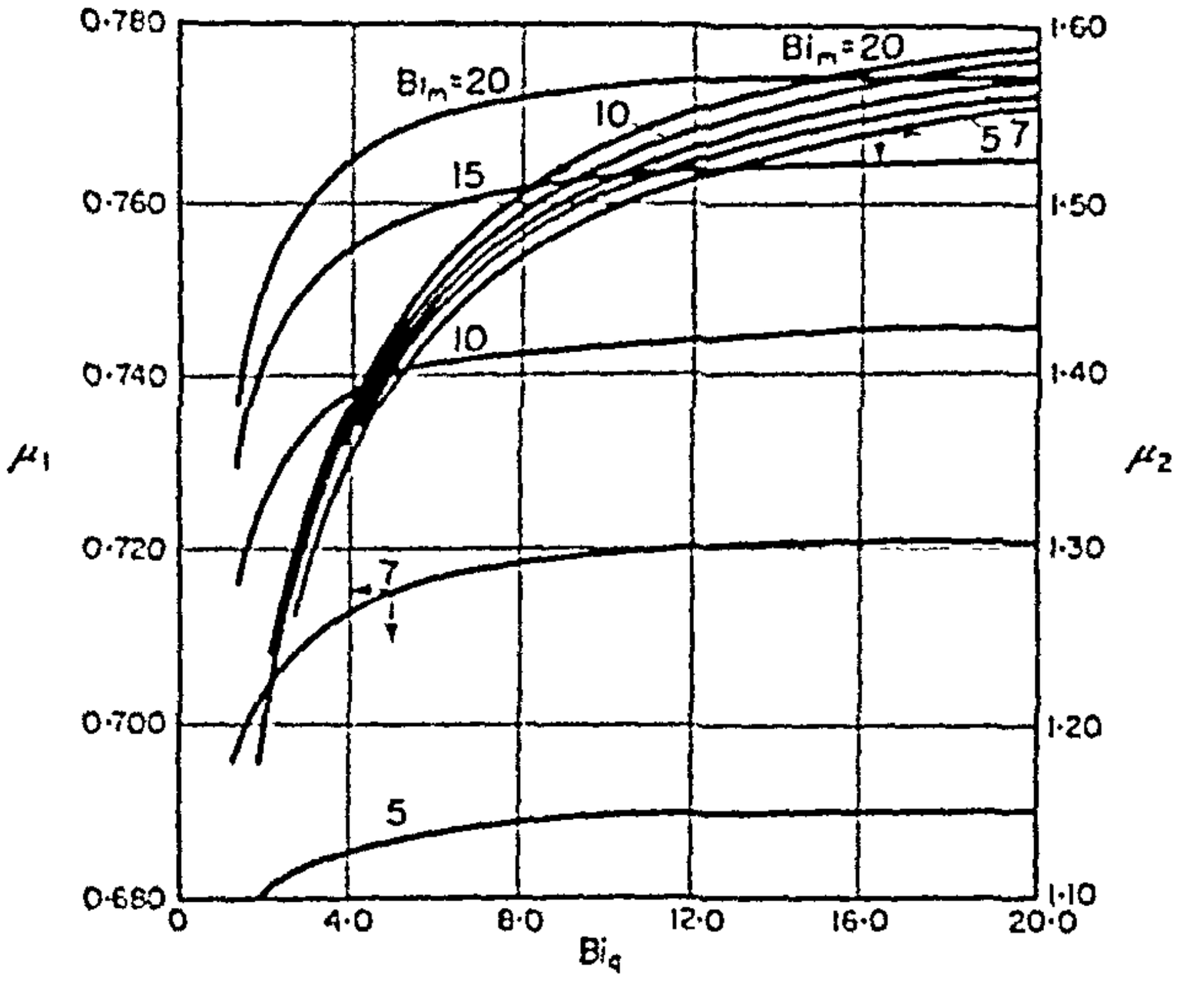


Fig. 5. Relation of the characteristic roots to Bi_m for an infinite plate ($Bi_q = 2$ to 20)

$$D_{n1} = \frac{\sin \nu_2 \mu_n}{\nu_2 \mu_n} C_{n1}, \; D_{n2} = \frac{\sin \nu_1 \mu_n}{\nu_1 \mu_n} C_{n2},$$

$$D_{n1}^* = \frac{\sin \nu_2 \mu_n}{\nu_2 \mu_n} (1 - \nu_2^2) C_{n1}^*, \; D_{n2}^* = \frac{\sin \nu_1 \mu_n}{\nu_1 \mu_n} (1 - \nu_1^2) C_{n2}^*. \qquad (1.39)$$

The calculated values of the coefficients D_n and D_n^* and their graphs with respect to the criteria Lu, Bi_q, Bi_m, ε and Pn are collected in Tables 7–12.

The non-dimensional rate of change of temperature and transfer potential of bound matter (local and mean) can be obtained by differentiating equations (1.28), (1.29), (1.36) and (1.37) with respect to the non-dimensional time, i.e.

$$\frac{dT}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} (C_{n2} \cos \nu_1 \mu_n x/R - C_{n1} \cos \nu_2 \mu_n x/R) \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.40)$$

TABLE 7. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF THE
(Ko = 1·2, Pn = 0·5, ε = 0·5,

Lu	C^*_{n1}		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·15	—0·1336	—0·01978	0·02312	1·258	—0·2226	—0·03298
0·3	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
0·5	—0·5840	—0·02162	0·09142	0·6064	—0·9732	—0·03606
0·7	—0·8483	—0·01546	0·01270	0·3146	—1·415	—0·02574
1·0	—1·135	—0·003402	0·1399	0·03438	—1·892	—0·00572
2·0	—1·452	0·04048	0·1118	—0·1828	—2·420	0·06738

TABLE 8. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF THE
(Lu = 0·3, ε = 0·5, Ko =

Bi_q	C_m		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1·0	—0·2986	—0·009234	1·348	—0·3906	—0·4974	—0·01539
5·0	—0·3102	—0·008446	0·1563	0·8598	—0·5168	—0·01409
7·0	—0·3088	—0·01090	0·09294	0·9122	—0·5148	—0·01816
10	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
15	—0·3074	—0·01556	0·01517	0·9524	—0·5122	—0·02594
20	—0·3068	—0·01634	0·002564	0·9630	—0·5116	—0·02274

TABLE 9. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF THE
(Lu = 0·3, ε = 0·5, Ko =

Bi_m	C_m		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·1	—2·558	0·1000	16,65	0·3122	—4·266	0·1667
0·5	—0·2500	0·02192	0·2408	0·5598	—0·4168	—0·03656
5	—0·2974	—0·03146	0·00369	0·9466	—0·4958	—0·05242
7	—0·3034	—0·02174	0·02590	0·9428	—0·5054	—0·03622
10	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
15	—0·3122	—0·009438	0·07318	0·9320	—0·5202	—0·01574
20	—0·3136	—0·006896	0·08182	0·9284	—0·5228	—0·01148

SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO Lu. $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_q = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·03852	2·096	0·1144	0·8651	0·0240	—0·1700
0·08130	1·564	0·2570	0·6500	0·9616	—0·1137
0·1524	1·011	0·4821	0·4232	0·974	—0·1274
0·2124	0·5246	0·6945	0·2213	0·9375	—0·0991
0·2334	0·0572	0·9009	0·02439	0·8435	—0·01538
0·1864	—0·3044	1·100	—0·1227	0·6463	0·132

SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO Bi_q 1·2, $\mathrm{Pn} = 0{\cdot}5$, $\mathrm{Bi}_m = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2·246	—0·6514	1·473	—0·3308	0·7652	0·08159
0·2606	1·434	0·3582	0·6291	0·9531	—0·1088
0·1550	1·521	0·2984	0·6491	0·9579	—0·1129
0·08130	*1·564*	*0·2570*	*0·6500*	*0·9616*	*—0·1137*
0·02526	1·588	0·2254	0·6472	0·9635	—0·1146
—0·004274	1·606	0·2084	0·6473	0·9643	—0·1150

SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO Bi_m 1·2, $\mathrm{Pn} = 0{\cdot}5$, $\mathrm{Bi}_q = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
27·74	0·5206	12·83	0·1939	—0·178	0·00417
0·4018	0·0346	0·4660	0·5304	1·104	—0·02065
0·006154	1·578	0·2202	0·6635	0·9910	—0·1087
0·04320	1·572	0·2384	0·6570	0·9751	—0·1120
0·08130	1·564	0·2570	0·6500	0·9616	—0·1137
0·1219	1·554	0·2776	0·6423	0·9502	—0·1143
0·1364	1·548	0·2835	0·6378	0·9410	—0·1144

TABLE 10. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF (Lu = 0·3, Ko = 1·2, Pn =

ε	C_m		C_{m2}		C_m^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·1667	—0·1173	—0·006474	0·08554	1·112	—0·5868	—0·03238
0·3333	—0·2190	—0 01140	0·06714	1·018	—0·5472	—0·02850
0·5000	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
0·6667	—0·3872	—0·01751	0·03478	0·8638	—0·4840	—0·02190
0·8333	—0·4592	—0·02002	0·02734	0·7946	—0·4592	—0·02002
1·0000	—0·5244	—0·01967	0·01669	0·7340	—0·4368	—0·01640

TABLE 11. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF (Lu = 0·3, Ko = 1·2, ε =

Pn	C_m		C_{m2}		C_m^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·1667	—0·3188	—0·005276	0·04618	0·9158	—0·5312	—0·008794
0·3333	—0·3132	—0·009060	0·04718	0·9288	—0·5220	—0·01509
0·5000	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
0·6667	—0·3028	—0·01809	0·04882	0·9524	—0·5048	—0·03016
0·8333	—0·2978	—0·02246	0·04830	0·9694	—0·4964	—0·03742

TABLE 12. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF (Lu = 0·3, Pn = 0·5, ε =

Ko	C_m		C_{m2}		C_m^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·4	—0·1172	—0·006408	0·01408	1·33	—0·5862	—0·03204
0·8	—0·2186	—0·01032	0·03190	1·029	—0·5466	—0·02580
1·2	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
1·6	—0·3876	—0·01630	0·06270	0·8658	—0·4844	—0·02036
2·0	—0·4592	—0·01827	0·07526	0·8072	—0·4592	—0·01827

THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO ε 0·5, $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_\varsigma = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·4276	5·562	0·1590	0·7636	0·990	—0·1433
0·1679	2·546	0·2128	0·7021	0·9753	—0·1255
0·08130	1·564	0·2570	0·6560	0·9616	—0·1137
0·04352	1·080	0·2979	0·6204	0·9484	—0·0119
0·02734	0·7946	0·3401	0·5540	0·9393	—0·09164
0·01391	0·6116	0·3738	0·5143	0·9301	—0·05296

THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO Pn 0·5, $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_\varsigma = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·07624	1·527	0·2620	0·6353	0·9045	—0·08952
0·07850	1·549	0·2590	0·6452	0·9336	—0·07795
0·08130	1·564	0·2570	0·6509	0·9516	—0·1137
0·08133	1·558	0·2532	0·6560	0·9595	—0·1509
0·05048	1·616	0·2493	0·6709	1·010	—0·1562

THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRASFER TO Ko 0·5, $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_\varsigma = 10$).

C^*_{n2}		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·07039	5·670	0·0·371	0·7859	0·9359	—0·1479
0·07982	2·574	0·1759	0·7148	0·9777	—0·1293
0·08130	1·564	0·2570	0·6560	0·9516	—0·1137
0·07842	1·082	0·03244	0·6091	0·9465	—0·1022
0·07526	0·8072	0·3850	0·5556	0·9338	—0·09302

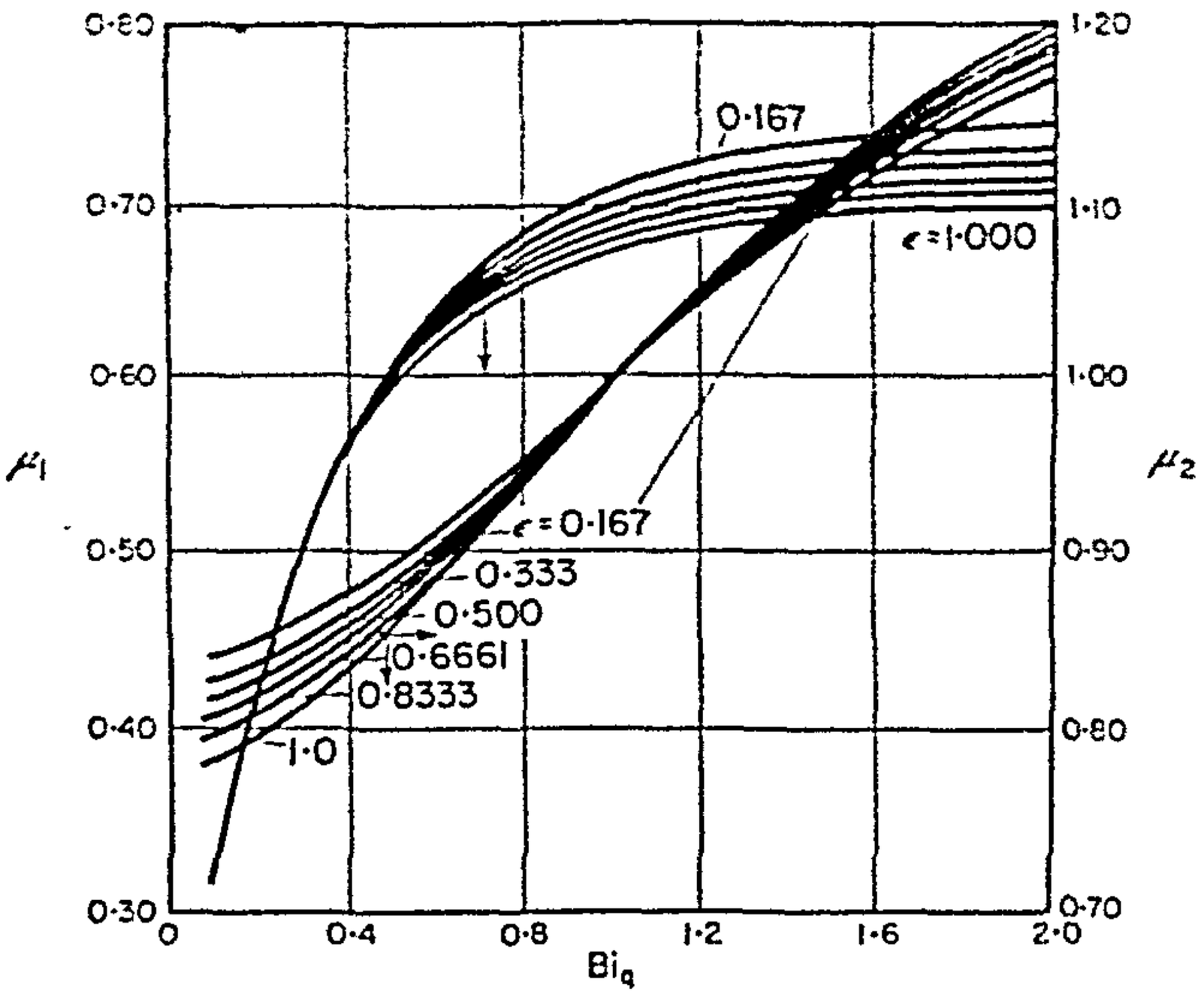


FIG. 6. Relation of the characteristic roots to ε for an infinite plate ($Bi_q = 0{\cdot}1$ to $2{\cdot}0$)

$$\frac{dT}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \tag{1.41}$$

$$\frac{d\,\Theta}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \left[C_{n1}^{*}(1-\nu_2^2)\cos\nu_2\,\mu_n\,x/R - \right.$$

$$\left. - C_{n2}^{*}(1-\nu_1^2)\cos\nu_1\,\mu_n\,x/R\right]\mu_n^2 \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \tag{1.42}$$

$$\frac{d\,\overline{\Theta}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^{*}\mu_n^2 \exp\left(-\mu_n^2\,\mathrm{Fo}\right). \tag{1.43}$$

Since $\mathrm{Fo} = a_q\tau/R^2$, the local and mean rates of heating and drying for a damp plate will be inversely proportional to the square of its characteristic dimension (half the thickness of the plate) and directly proportional to the coefficient of temperature conductivity.

(b) *Sphere. Constant initial conditions* ($\Gamma = 2$, $\xi = r$, $f(r) = t_0 = \text{const.}$, $\varphi(r) = \theta_0 = \text{const.}$). Let us introduce the symbols

$$L[rt(r,\tau)] = \int_0^{\infty} rt(r,\tau)\,e^{-s\tau}\,d\tau = \vartheta_L(r,s) = rt_L(r,s), \tag{1.44}$$

$$L[r\,\theta(r,\tau)] = \int_0^\infty r\,\theta(r,\tau)\,e^{-s\tau}\,d\tau$$

$$= u_L(r,s) = r\,\theta_L(r,s)\,. \qquad (1.45)$$

Applying the Laplace transformations (1.44), (1.45) to the equations (I–II) we shall have

$$\vartheta_L''(r,s) - \frac{s}{a_1}\vartheta_L(r,s) +$$

$$+ \frac{\varepsilon\varrho}{a_q c}\,s u_L(r,s) - \frac{\varepsilon\varrho}{a_q c}\,r\,\theta_0 + \frac{r t_0}{a_q} = 0\,, \qquad (1.46)$$

$$u_L''(r,s) - \frac{s}{a_m}\,u_L(r,s) + \delta\vartheta_L''(r,s) + \frac{r\,\theta_0}{a_m} = 0\,. \qquad (1.47)$$

From equation (1.46) we can obtain $u_L(r,s)$ and its second derivative $u_L''(r,s)$. We substitute the expressions thus found

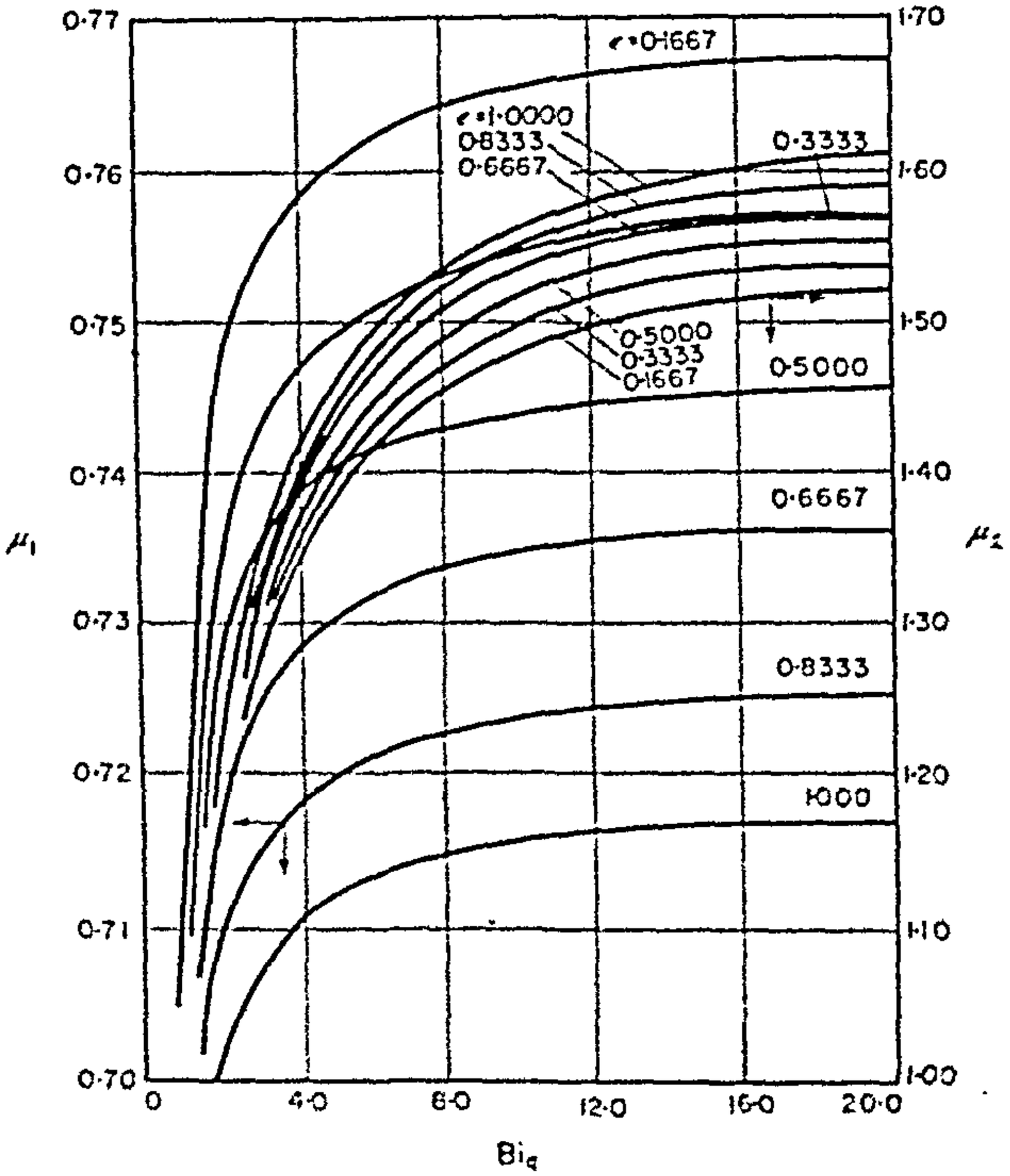


FIG. 7. Relation of the characteristic roots to ε for an infinite plate ($Bi_q = 1$ to 20)

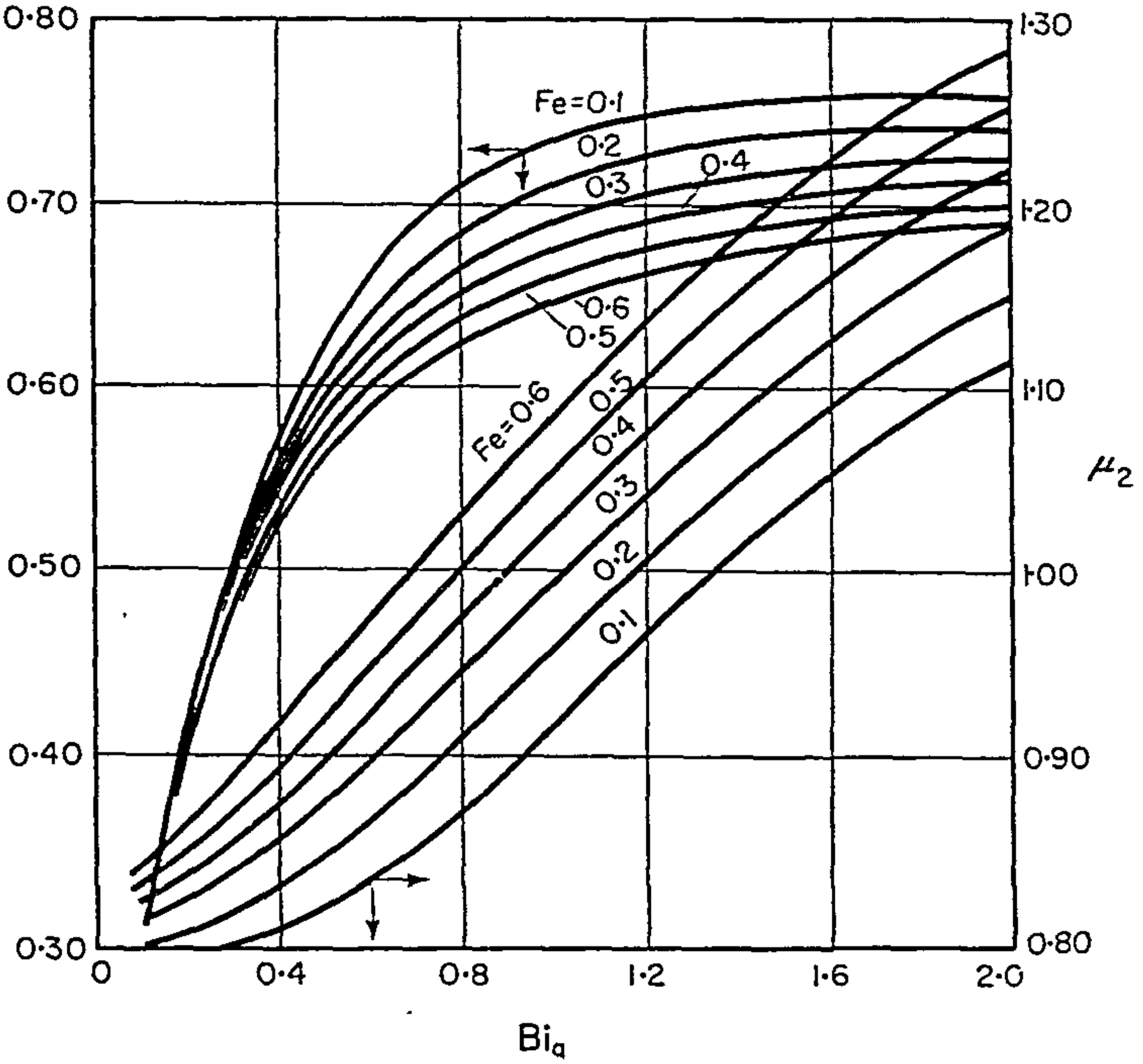


FIG. 8. Relation of the characteristic roots to Fe for an infinite plate ($Bi_q = 0{\cdot}1$ to $2{\cdot}0$)

in (1.47). After transformation we obtain for the modified transform of the temperature $\vartheta_L(r, s)$ the non-homogeneous ordinary differential equation of the fourth order

$$\vartheta_L^{IV}(r, s) - \left(\frac{s}{a_q} + \frac{\varepsilon\varrho\delta S}{a_q c} + \frac{s}{a_m}\right)\vartheta_L''(r, s) + \\ + \frac{s^2}{a_q a_m}\vartheta_L(r, s) - \frac{s r t_0}{a_q a_m} = 0\,. \qquad (1.48)$$

The general integral of equation (1.48) can be written:

$$v_L(r, s) = \frac{r t_0}{s} + C_1 e^{\sqrt{\left(\frac{s}{a_q}\right)} v_1 r} + \\ + C_2 e^{\sqrt{\left(\frac{s}{a_q}\right)} v_2 r} + C_3 e^{-\sqrt{\left(\frac{s}{a_q}\right)} v_1 r} + C_4 e^{-\sqrt{\left(\frac{s}{a_q}\right)} v_2 r}, \qquad (1.49)$$

where C_k ($k = 1, 2, 3, 4$) are arbitrary constants and v_j is expressed by equation (1.5).

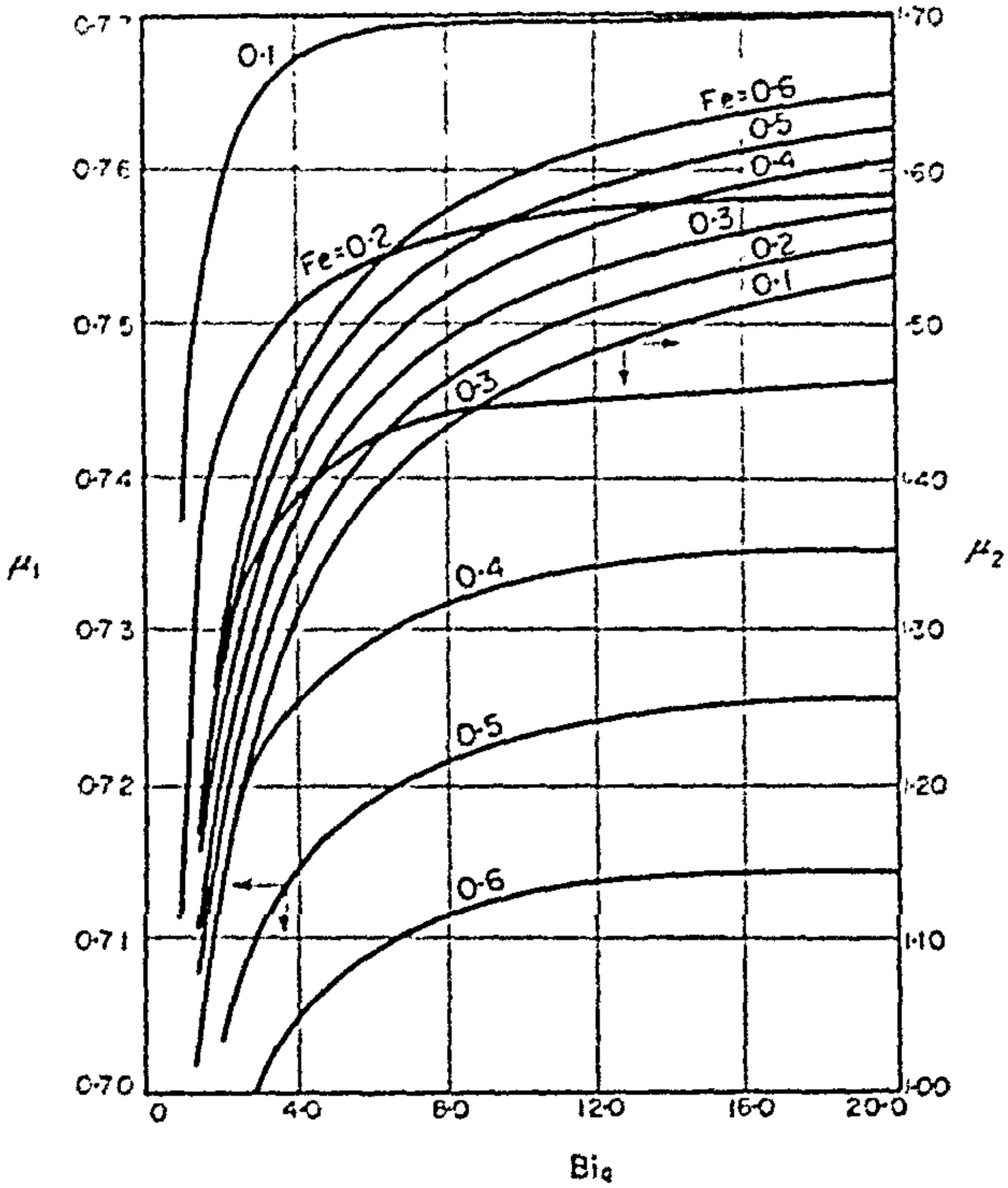


FIG. 9. Relation of the characteristic roots to Fo for an infinite plate ($Bi_q = 1{\cdot}0$ to 20)

The solution for $u_L(r, s)$ will be found by substituting $\vartheta_L''(r, s)$ and $\vartheta_L(r, s)$ in equation (1.46). We have

$$u_L(r,s) = \frac{r\,\vartheta_0}{s} + C_1 \frac{c}{\varepsilon\varrho}(1-\nu_1^2)\,e^{\sqrt{\left(\frac{s}{a_t}\right)}\nu_1 r} +$$

$$+ C_2 \frac{c}{\varepsilon\varrho}(1-\nu_2^2)\,e^{\sqrt{\left(\frac{s}{a_t}\right)}\nu_2 r} +$$

$$+ C_3 \frac{c}{\varepsilon\varrho}(1-\nu_1^2)\,e^{-\sqrt{\left(\frac{s}{a_t}\right)}\nu_1 r} + C_4 \frac{c}{\varepsilon\varrho}(1-\nu_2^2)\,e^{-\sqrt{\left(\frac{s}{a_t}\right)}\nu_2 r}. \quad (1.50)$$

The constants C_k are determined by the boundary conditions and the conditions of symmetry (III–V). The latter will

be, after the application of the Laplace transformation:

$$t'_L(0, s) \text{ and } \theta'_L(0, s) = 0.$$

Since

$$r^2 \frac{dt_L}{dr} = \frac{d\vartheta_L}{dr} r - \vartheta_L \text{ and } t'_L(0, \tau) < \infty,$$

$$\lim_{r \to 0} \left[r^2 \frac{dt_L(r, s)}{dr} \right] = \lim_{r \to 0} \left[r \frac{d\vartheta_L(r, s)}{dr} - \vartheta_L(r, s) \right] = 0.$$

The condition of symmetry for the modified temperature can thus be written in the form

$$\vartheta_L(0, s) = 0 . \tag{1.51}$$

The condition for the modified potential of mass transfer can, in an analogous way, be written

$$u_L(0, s) = 0 . \tag{1.52}$$

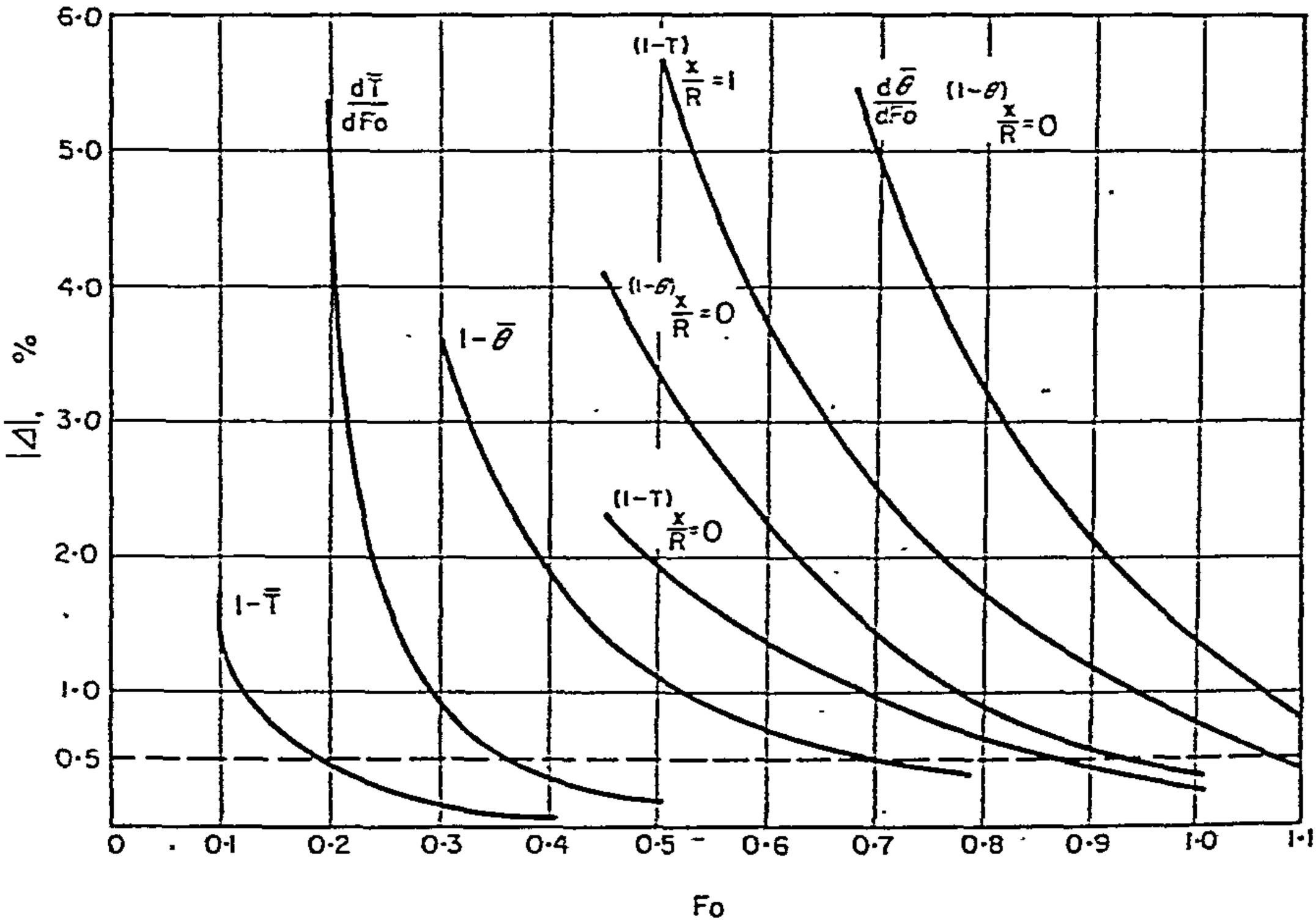


FIG. 10. The error introduced into the determination $1 - T$, $1 - \Theta$, $1 - \bar{T}$, $1 - \bar{\Theta}$, $d\bar{T}/d\mathrm{Fo}$, and $d\bar{\Theta}/d\mathrm{Fo}$ by neglecting the terms of the infinite series beginning from the third

It follows from (1.51) and (1.52) that $C_3 = -C_1$, $C_4 = -C_2$. Thus equations (1.49) and (1.50) can be somewhat simplified:

$$\vartheta_L(r, s) - \frac{rt_0}{s} = B_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r + \\ + B_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r , \tag{1.53}$$

$$u_L(r, s) - \frac{r\,\theta_0}{s} = B_1 \frac{c}{\varepsilon\varrho} (1 - \nu_1^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r + \\ + B_2 \frac{c}{\varepsilon\varrho} (1 - \nu_2^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r , \tag{1.54}$$

or

$$t_L(r, s) - \frac{t_0}{s} = B_1 \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r}{r} + \\ + B_2 \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r}{r} , \tag{1.55}$$

$$\theta_L(r, s) - \frac{\theta_0}{s} = B_1 \frac{c}{\varepsilon\varrho} (1 - \nu_1^2) \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r}{r} + \\ + B_2 \frac{c}{\varepsilon\varrho} (1 - \nu_2^2) \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r}{r} . \tag{1.56}$$

Here $B_1 = 2C_1$ and $B_2 = 2C_2$ are new constants with respect to the coordinates. We will determine B_1 and B_2 from boundary conditions (III–IV) which, *after* the application of the Laplace transformation, can be written:

$$-\lambda t'_L(R, s) + \alpha \left[\frac{t_c}{s} - t_L(R, s)\right] - \\ - (1 - \varepsilon) \varrho\beta \left[\theta_p(R, s) - \frac{\theta_p}{s}\right] = 0 . \tag{1.57}$$

$$\lambda_m \theta'_L(R, s) + \lambda_m \delta t'_L(R, s) + \\ + \beta \left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0 . \tag{1.58}$$

Let us adjust solutions (1.55) and (1.56) to the boundary conditions (1.57) and (1.58). After simplification, the system of equations for B_1 and B_2 can be written in the following form:

$$B_1 \widetilde{Q}_1 + B_2 \widetilde{Q}_2 - \frac{t_c - t_0}{\varepsilon} + (1-\varepsilon)\frac{\varrho\beta}{\alpha}\,\frac{\theta_0 - \theta_p}{\varepsilon} = 0\,,$$

$$B_1 \widetilde{P}_1 + B_2 \widetilde{P}_2 + \frac{\mathrm{Fe}}{\delta}\,\frac{\theta_0 - \theta_p}{\varepsilon} = 0. \qquad (1.59)$$

Hence we can determine the coefficients B_j. We have

$$B_1 = -\frac{(t_c - t_0)\,\widetilde{P}_2 + \frac{\varepsilon\varrho}{c}(\theta_0 - \theta_p)(\widetilde{Q}_2 - K_1\widetilde{P}_2)}{\varepsilon(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)}\,,$$

$$B_2 = \frac{(t_c - t_0)\,\widetilde{P}_1 + \frac{\varepsilon\varrho}{c}(\theta_0 - \theta_p)(\widetilde{Q}_1 - K_1\widetilde{P}_1)}{\varepsilon(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)}\,,$$

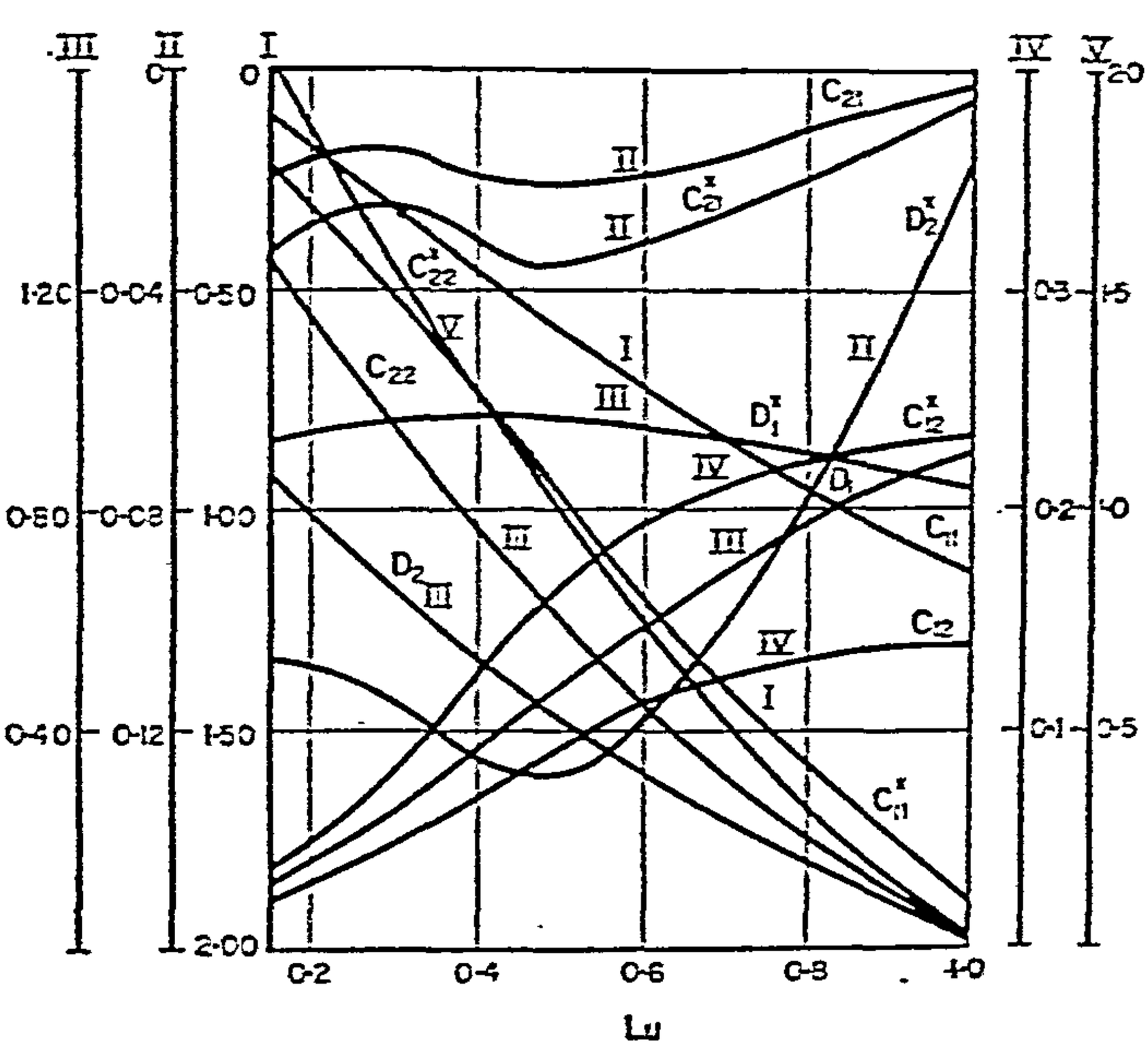


FIG. 11. Relation of the coefficients of heat and Mass transfer to the criterion Lu

where

$$\widetilde{Q}_j = \frac{1}{\mathrm{Bi}_q}\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R\,\frac{\cosh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R}{R} +$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}-(1-\nu_j^2)\,K_1\right]\frac{\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R}{R}, \qquad (1.60)$$

$$\widetilde{P}_j = [(1-\nu_j^2)+\mathrm{Fe}]\,\frac{1}{\mathrm{Bi}_m}\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R\,\frac{\cosh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R}{R} +$$

$$+\left[(1-\nu_j^2)-\frac{(1-\nu_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\frac{\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_j R}{R}. \qquad (1.61)$$

Thus the solutions for the modified transformations of the transfer potentials (1.53) and (1.54) will be as follows:

$$\vartheta_L(r,s)-\frac{rt_0}{s}$$

$$=(t_c-t_0)\left\{\frac{\widetilde{P}_1+\varepsilon\,\mathrm{Ko}\,(\widetilde{Q}_1-K_1\widetilde{P}_1)}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_2 r-\right.$$

$$\left.-\frac{\widetilde{P}_2+\varepsilon\,\mathrm{Ko}(\widetilde{Q}_2-K_1\widetilde{P}_2)}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_1 r\right\}$$

$$=(t_c-t_0)\,\frac{\Psi_1(s)}{\Psi_1(s)}, \qquad (1.62)$$

$$\frac{r\theta_0}{s}-u_L(r,s)=(\theta_0-\theta_p)\left\{\frac{\widetilde{Q}_2+\frac{1}{\varepsilon\,\mathrm{Ko}}\widetilde{P}_2-K_1\widetilde{P}_2}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\times\right.$$

$$\times(1-\nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_1 r-\frac{\widetilde{Q}_1+\frac{1}{\varepsilon\,\mathrm{Ko}}\widetilde{P}_1-K_1\widetilde{P}_1}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\times$$

$$\left.\times(1-\nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,\nu_2 r\right\}=(\theta_0-\theta_p)\,\frac{\Psi_2(s)}{\Psi_1(s)}. \qquad (1.63)$$

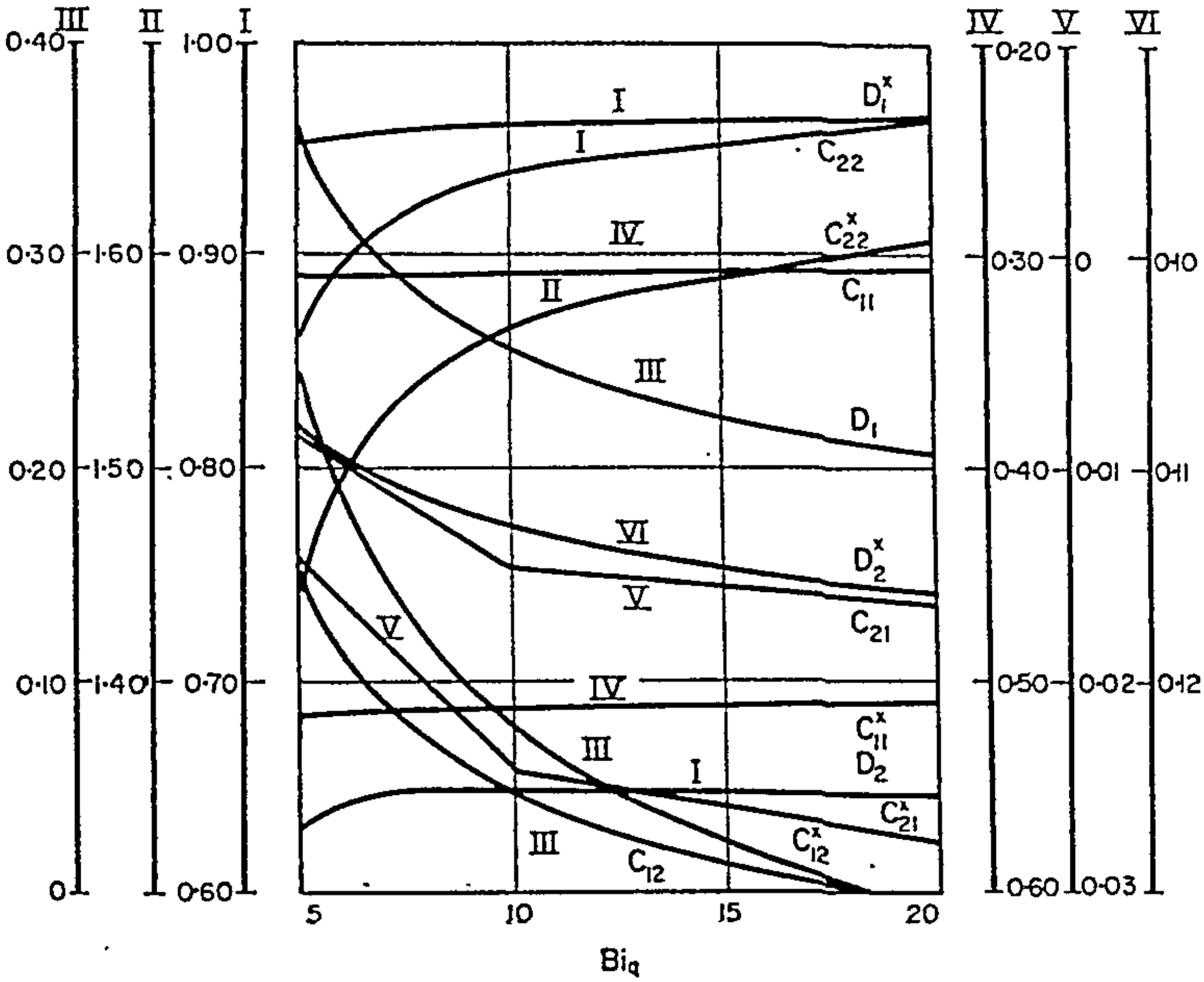


FIG. 12. Relation of the coefficients of heat and mass transfer to Bi_q

The numerator and denominator in (1.62) and (1.63) are not generalized polynomials with respect to s, but they can be made such by dividing both parts, i.e. $\Phi_1(s)$ and $\psi_1(s)$, $\Phi_2(s)$ and $\psi_1(s)$ by $\sqrt{(s/a_q)}$. Analysis of these equations shows also that the inversion of the transform to original, except for the null root ($s = 0$), can be carried out by the ordinary expansion theorem, that is, without transforming $\Phi_1(s)$ or $\Phi_2(s)$ into the generalized polynomial $\Phi_{01}(s)$ or $\Phi_{02}(s)$ and $\psi_1(s)$ into $\psi_0(s)$.

Let us find the roots of $\psi_1(s)$:

$$\psi_1(s) = s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2) = 0 .$$

We have

(a) $s = s_0 = 0$ (null root),

(b) $s = s_n$, where s_n satisfies the equation

$$\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2 = 0 . \qquad (1.64)$$

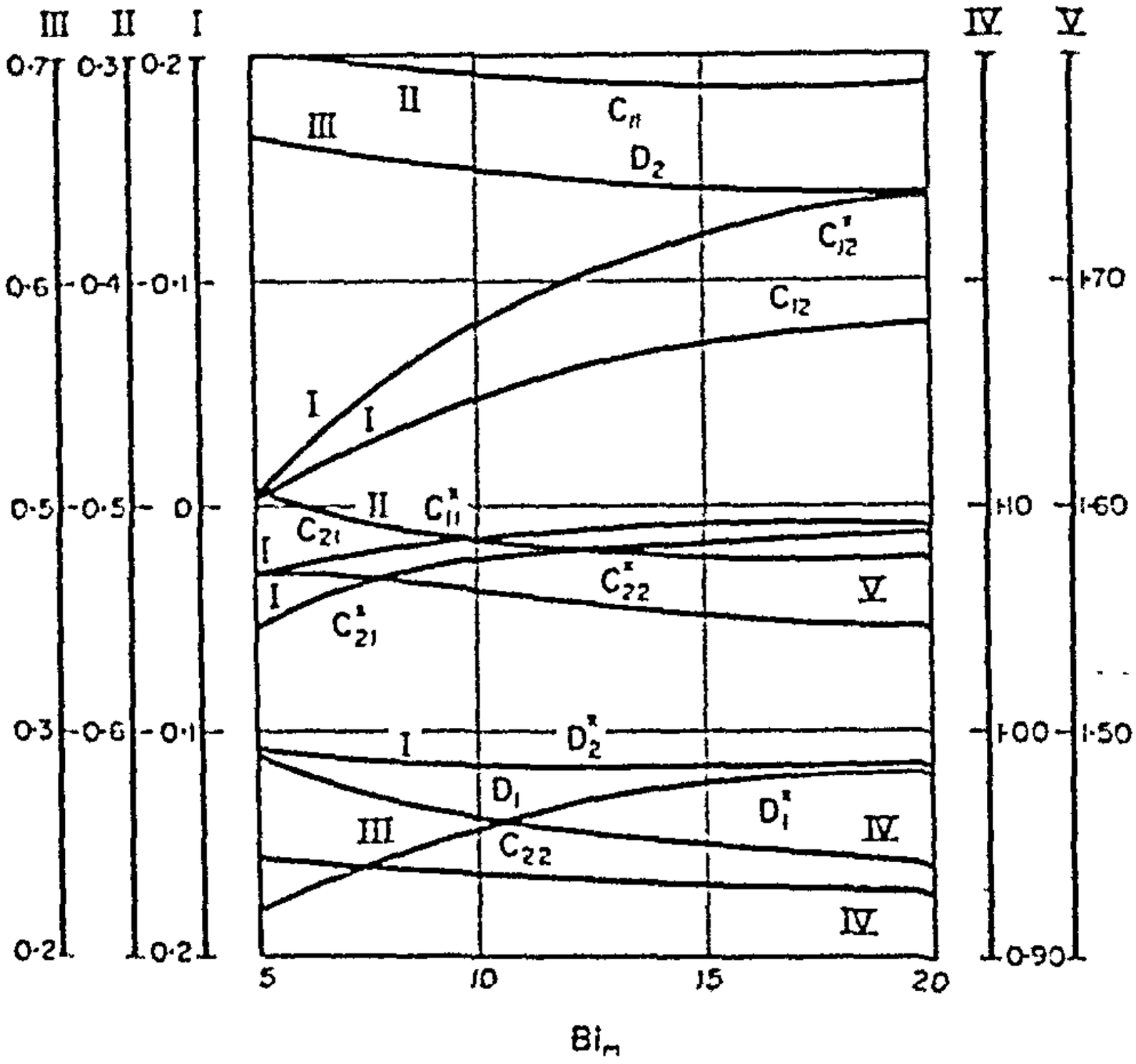


FIG. 13. Relation of the coefficients of heat and mass transfer to Bi_m

Let us replace the hyperbolic functions in equation (1.64) by trigonometrical functions and denote $i\sqrt{s/a_q}\,R$ by μ; then we have the following expression for s_n:

$$s_n = -\frac{a_q \mu_n^2}{R^2},$$

where μ_n is defined by the equation

$$Q_{n2} P_{n1} - Q_{n1} P_{n2} = 0. \qquad (1.65)$$

Here

$$Q_{nj} = \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n +$$

$$+ \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n, \qquad (1.66)$$

$$P_{nj} = [(1 - \nu_j^2) + \mathrm{Fo}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \cos \nu_j \mu_n +$$

$$+ \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \mathrm{Fo}}{\mathrm{Bi}_m}\right] \sin \nu_j \mu_n. \qquad (1.67)$$

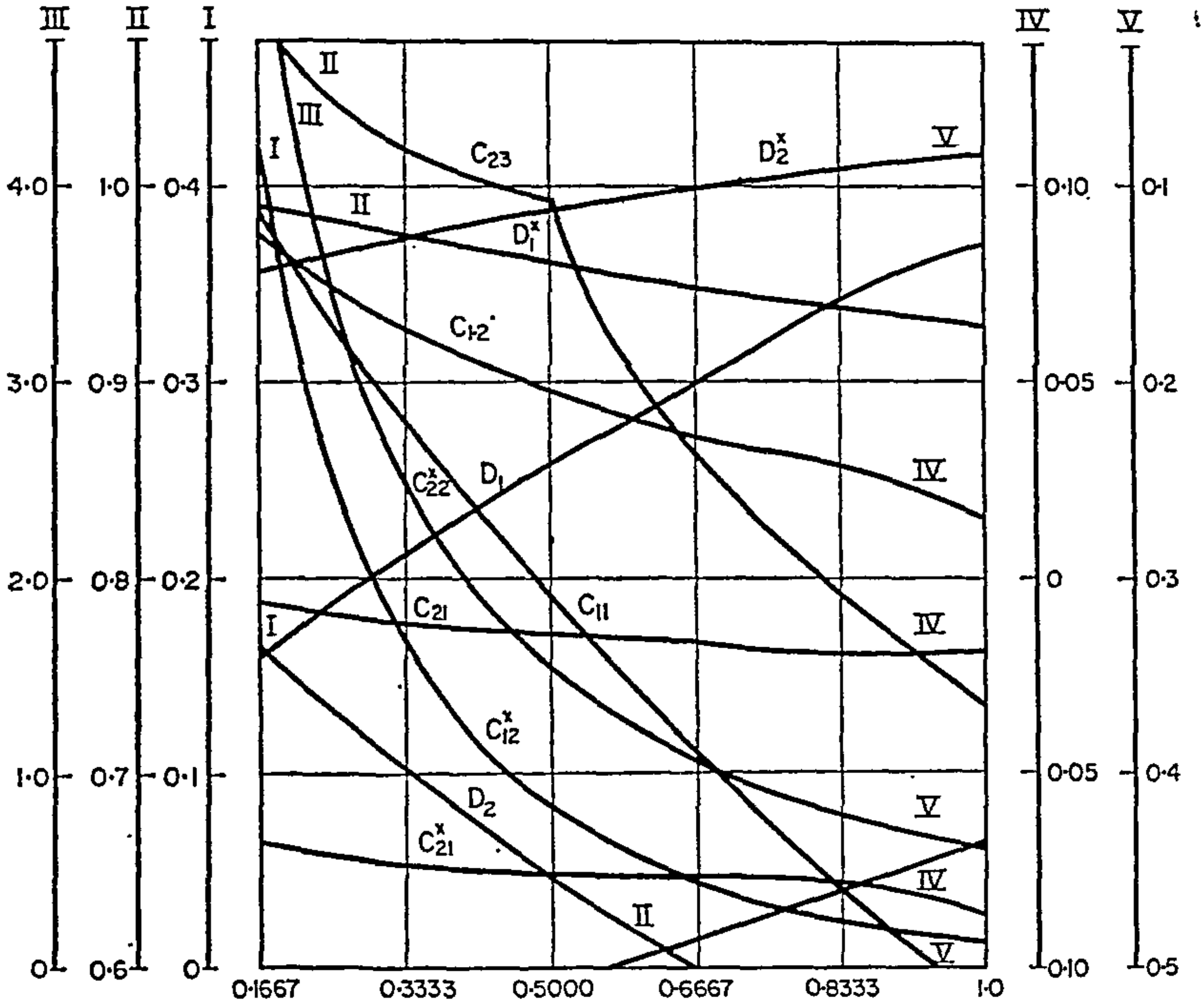


FIG. 14. Relation of the coefficients of heat and mass transfer to ε

Thus the inversion of the transform may be written in the form

$$L^{-1}\left[\frac{\Phi_1(s)}{\psi_1(s)}\right] = \frac{\Phi_{01}(0)}{\psi_0'(0)} + \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi_1'(s_n)} \exp(s_n \tau), \qquad (1.68)$$

$$L^{-1}\left[\frac{\Phi_2(s)}{\psi_1(s)}\right] = \frac{\Phi_{02}(0)}{\psi_0'(0)} + \sum_{n=1}^{\infty} \frac{\Phi_2(s_n)}{\psi_1'(s_n)} \exp(s_n \tau). \qquad (1.69)$$

To determine the null term we transform equation (1.68) in such a way that it becomes the ratio of two generalized polynomials. The latter will be:

$$\Phi_{01}(s) = \left\{ \left[\frac{(1-\nu_1^2) + \mathrm{Fe}}{\mathrm{Bi}_m} \nu_1 \left(1 + \frac{1}{2!} \frac{s}{a_q} \nu_1^2 R^2 + \ldots\right) + \right.\right.$$

$$\left. + \left[(1-\nu_1^2) - \frac{(1-\nu_1^2) + \mathrm{Fe}}{\mathrm{Bi}_m}\right] \left(\nu_1 + \frac{1}{3!} \frac{s}{a_q} \nu_1^3 R^2 + \ldots\right)\right] +$$

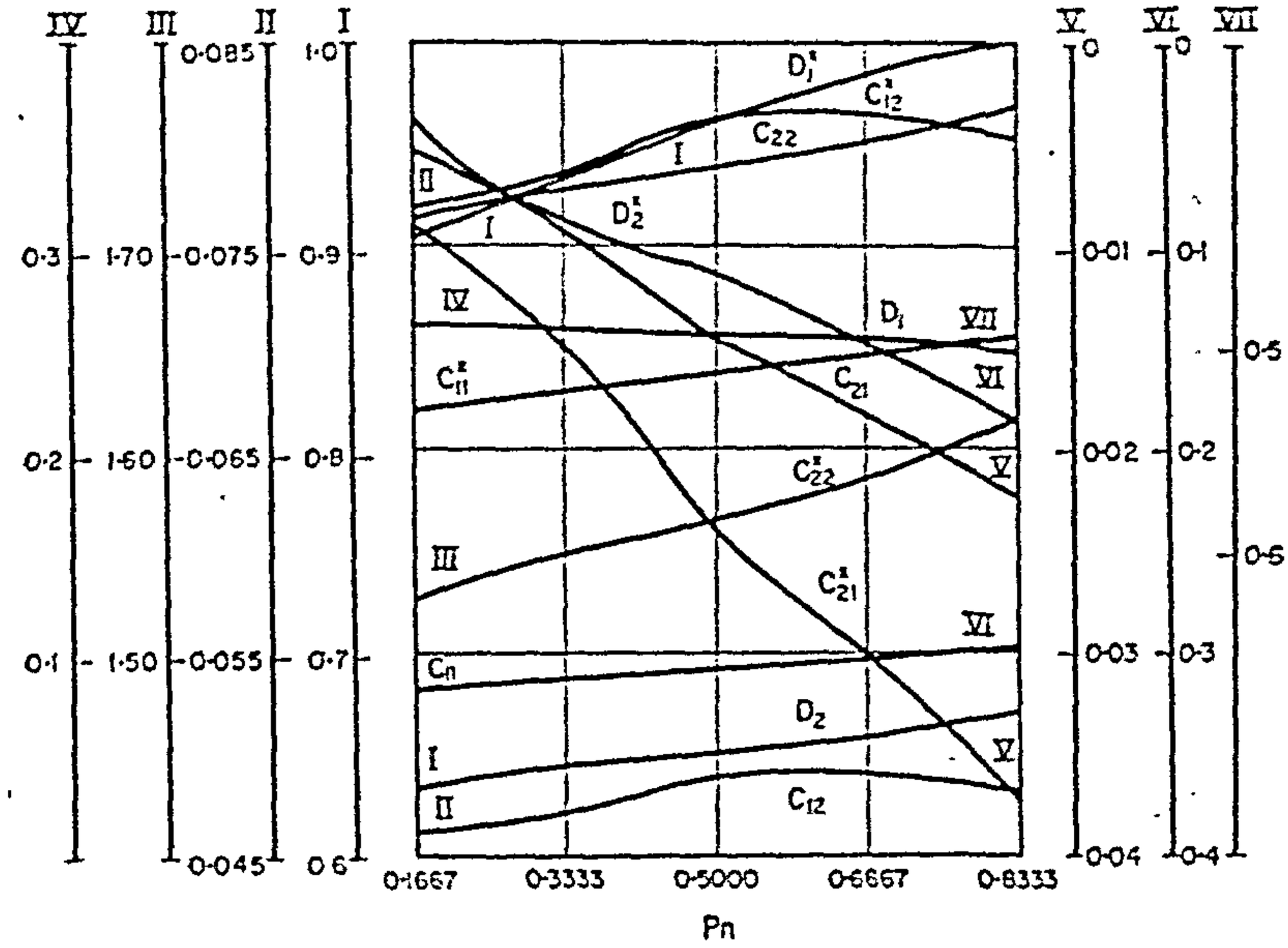


FIG. 15. Relation of the coefficients of heat and mass transfer to Pn

$$+\varepsilon\,\mathrm{Ko}\left[\frac{\nu_1}{\mathrm{Bi}_q}\left(1+\frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_1^2R^2+\ldots\right)+\right.$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}+(1-\nu_1^2)\,K_1\right]\left(\nu_1+\frac{1}{3!}\,\nu_1^3R^2\,\frac{\varepsilon}{a_q}+\ldots\right)-$$

$$-K_1\left\{\frac{(1-\nu_1^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\,\nu_1\left(1+\frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_1^2R^2+\ldots\right)+\right.$$

$$+\left[(1-\nu_1^2)-\frac{(1-\nu_1^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\right]\times$$

$$\left.\left.\times\left(\nu_1+\frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_1^3R^2+\ldots\right)\right)\right]\Bigg\}\times$$

$$\times\left(\nu_2 r+\frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_2^3r^3+\ldots\right)-$$

$$-\left\{\left[\frac{(1-\nu_2^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\,\nu_2\left(1+\frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_2^2R^2+\ldots\right)+\right.\right.$$

$$\left.+\left[(1-\nu_2^2)-\frac{(1-\nu_2^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\right]\left(\nu_2+\frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_2^3R^2+\ldots\right)\right]+$$

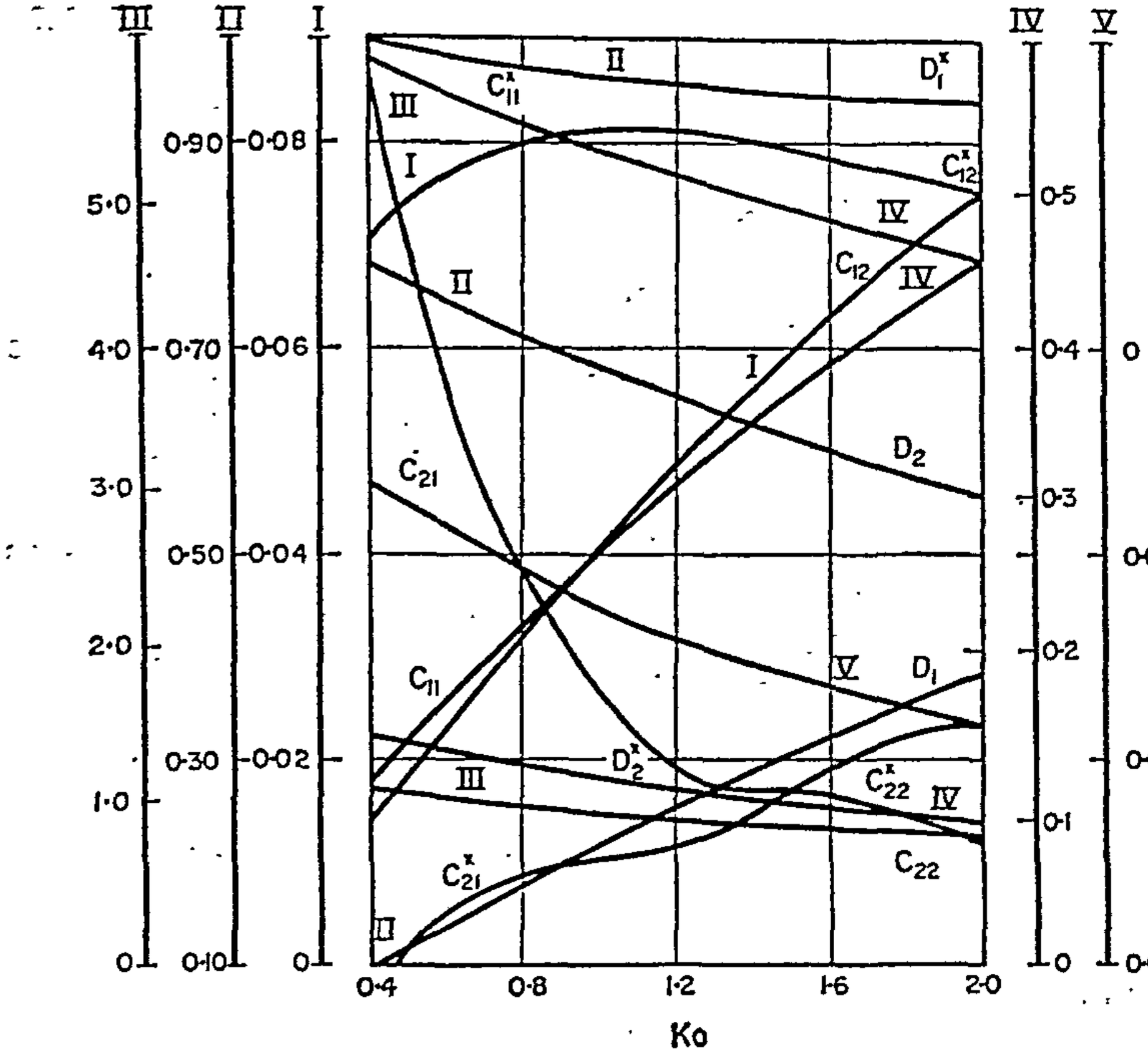


FIG. 16. Relation of the coefficients of heat and mass transfer to the criterion Ko

$$+\varepsilon\,\mathrm{Ko}\left[\frac{\nu_2}{\mathrm{Bi}_q}\left(1+\frac{1}{2!}\,\frac{s}{a_q}\nu_2^2R^2+\ldots\right)+\right.$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}+(1-\nu_2^2)\,K_1\right]\left(\nu_2+\frac{1}{3!}\nu_2^3R^2\frac{s}{a_q}+\ldots\right)-$$

$$-K_1\left(\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\nu_2\left(1+\frac{1}{2!}\,\frac{s}{a_q}\nu_2^2R^2+\ldots\right)+\right.$$

$$\left.\left.+\left[(1-\nu_2^2)-\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\left(\nu_2+\frac{1}{3!}\,\frac{s}{a_q}\nu_2^3R^2+\ldots\right)\right)\right]\Bigg\}\times$$

$$\times\left(\nu_1\tau+\frac{1}{3!}\,\frac{s}{a_q}\nu_1^3\tau^3+\ldots\right),$$

$$\psi_0(s)=s\left[\left\{\frac{\nu_2}{\mathrm{Bi}_q}\left(1+\frac{1}{2!}\,\frac{s}{a_q}\nu_2^2R^2+\ldots\right)+\right.\right.$$

$$\left.+\left[1-\frac{1}{\mathrm{Bi}_q}+K_1(1-\nu_2^2)\right]\left(\nu_2+\frac{1}{3!}\nu_2^3R^2\frac{s}{a_q}+\ldots\right)\right\}\times$$

$$\times \left\{ \frac{(1-\nu_1^2)+\mathrm{Fo}}{\mathrm{Bi}_m} \nu_1 \left(1 + \frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_1^2 R^2 + \ldots\right) + \right.$$

$$\left. + \left[(1-\nu_1^2) - \frac{(1-\nu_1^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\right]\left(\nu_1 + \frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_1^3 R^2 + \ldots\right)\right\} -$$

$$- \left\{ \frac{\nu_1}{\mathrm{Bi}_q}\left(1 + \frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_1^2 R^2 + \ldots\right) + \left[1 - \frac{1}{\mathrm{Bi}_q} + K_1(l-\nu_1^2)\right] \times \right.$$

$$\times \left.\left(\nu_1 + \frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_1^3 R^2 + \ldots\right)\right\} \times$$

$$\times \left\{ \frac{(1-\nu_2^2)+\mathrm{Fo}}{\mathrm{Bi}_m} \nu_2 \left(1 + \frac{1}{2!}\,\frac{\varepsilon}{a_q}\nu_2^2 R^2 + \ldots\right) + \right.$$

$$\left. \left. + \left[(1-\nu_2^2) - \frac{(1-\nu_2^2)+\mathrm{Fo}}{\mathrm{Bi}_m}\right]\left(\nu_2 + \frac{1}{3!}\,\frac{\varepsilon}{a_q}\nu_2^3 R^2 + \ldots\right)\right\}\right].$$

Thus the null term of equation (1.68) will be written thus:

$$\lim_{\varepsilon \to 0} \frac{\Phi_{01}(\varepsilon)}{\Psi_0'(\varepsilon)} = r\,.$$

For the remaining roots we have

$$\psi_1'(\varepsilon_n) = \frac{\mu_n}{2R^2}\,\psi_n\,,$$

where

$$\psi_n = \nu_1 P_{n2} A_{n1} + \nu_2 Q_{n1} B_{n2} - \nu_2 P_{n1} A_{n2} - \nu_1 Q_{n2} B_{n1}\,, \quad (1.70)$$

$$A_{nj} = [1 + (1-\nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n\,, \quad (1.71)$$

$$B_{nl} = (1-\nu_j^2) \cos \nu_j \mu_n - [(1-\nu_j^2) + \mathrm{Fo}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n\,. \quad (1.72)$$

The term corresponding to the null roots of equation (1.68) will be

$$\sum_{n=1}^{\infty} \frac{\Phi_1(\varepsilon_n)}{\Psi_1'(\varepsilon_n)}\, e^{\varepsilon_n \tau}$$

$$= - \sum_{n=1}^{\infty} \left(C_{n1} R \sin \nu_2 \mu_n \frac{r}{R} - C_{n2} R \sin \nu_1 \mu_n \frac{r}{R}\right) \exp(-\mu_n^2 \mathrm{Fo}),$$

$$C_{nj} = 2\,\frac{P_{nj} + \varepsilon\,\mathrm{Ko}\,(Q_{nj} - K_1 P_{nj})}{\mu_n \psi_n}\,. \quad (1.73)$$

Thus the equation for the non-stationary heat transfer potential field has the form

$$\frac{rt(r,\tau)-rt_0}{t_c-t_0}=$$

$$=r-\sum_{n=1}^{\infty}\left(C_{n1}R\sin\nu_2\mu_n\frac{r}{R}-C_{n2}R\sin\nu_1\mu_n\frac{r}{R}\right)\exp\left(-\mu_n^2\mathrm{Fo}\right). \quad (1.74)$$

The method of obtaining the inversion of the transform (1.74) and (1.69) is exactly analogous to the above. In this case calculation gives

$$\frac{r\theta_0-r\theta(r,\tau)}{\theta_0-\theta_p}=r-\sum_{n=1}^{\infty}\left[C_{n2}^*(1-\nu_1^2)R\sin\nu_1\mu_n\frac{r}{R}-\right.$$

$$\left.-C_{n1}^*(1-\nu_2^2)R\sin\nu_2\mu_n\frac{r}{R}\right]\exp\left(-\mu_n^2\mathrm{Fo}\right), \quad (1.75)$$

$$C_{nj}^*=\frac{1}{\varepsilon\mathrm{Ko}}C_{nj}. \quad (1.76)$$

Thus we obtain as the final form of the solution of the system of differential equations for the heat and mass transfer in a sphere, with boundary conditions of the third kind and constant initial conditions, collecting together equations (1.65)–(1.67) and (1.70)–(1.76):

$$T=\frac{t(r,\tau)-t_0}{t_c-t_0}=1-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\dfrac{r}{R}}{\dfrac{r}{R}}-C_{n2}\frac{\sin\nu_1\mu_n\dfrac{r}{R}}{\dfrac{r}{R}}\right)\times$$

$$\times\exp\left(-\mu_n^2\mathrm{Fo}\right), \quad (1.77)$$

$$\Theta=\frac{\theta_0-\theta(r,\tau)}{\theta_0-\theta_p}=1-\sum_{n=1}^{\infty}\left[C_{n2}^*(1-\nu_1^2)\frac{\sin\nu_1\mu_n\dfrac{r}{R}}{\dfrac{r}{R}}-\right.$$

$$\left.-C_{n1}^*(1-\nu_2^2)\frac{\sin\nu_2\mu_n\dfrac{r}{R}}{\dfrac{r}{R}}\right]\exp\left(-\mu_n^2\mathrm{Fo}\right), \quad (1.78)$$

where

$$C_{nj} = 2\,\frac{P_{nj} + \varepsilon\,\mathrm{Ko}(Q_{nj} - K_1 P_{nj})}{\mu_n Y_n},$$

$$C^*_{nj} = \frac{1}{\varepsilon\,\mathrm{Ko}}\,C_{nj},$$

$$\nu_j^2 = \frac{1}{2}\left\{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j\sqrt{\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}\right]}\right\},$$

$$\psi_n = \nu_1 P_{n2} A_{n1} + \nu_2 Q_{n1} B_{n2} - \nu_2 P_{n1} A_{n2} - \nu_1 Q_{n2} B_{n1},$$

$$A_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - [(1 - \nu_j^2) + \mathrm{Fe}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n,$$

$$Q_{nj} = \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n +$$

$$+ \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n,$$

$$P_{nj} = [(1 - \nu_j^2) + \mathrm{Fe}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \cos \nu_j \mu_n +$$

$$+ \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m}\right] \sin \nu_j \mu_n.$$

μ_n are the roots of the characteristic equation (1.65)

$$Q_{n2} P_{n1} - Q_{n1} P_{n2} = 0.$$

This equation can be written thus:

$$\frac{M}{N} = \frac{\mu}{\mathrm{Bi}_q - 1} \tag{1.79}$$

where M and N denote the following expressions:

$$M = P_1 \mu \sin \nu_2 \mu - P_2 \mu \sin \nu_1 \mu,$$

TABLE 13. ROOTS OF THE CHARACTERISTIC EQUATION

$$\Phi(\mu) = \frac{1}{\mathrm{Bi}_q - 1}\,\mu \text{ FOR A SPHERE.}$$

Bi_q	μ_1	μ_2	μ_3	μ_4
	$\mathrm{Bi}_q < 1{\cdot}0$			
0·10	0·519	2·074	3·490	5·000
0·15	0·654	2·083	3·491	5·016
0·20	0·748	2·091	3·491	5·024
0·25	0·827	2·100	3·492	5·030
0·30	0·900	2·108	3·492	5·033
0·35	0·962	2·115	3·493	5·036
0·40	1·026	2·124	3·493	5·040
0·45	1·077	2·134	3·493	5·045
0·50	1·122	2·144	3·494	5·050
0·55	1·167	2·155	3·494	5·056
0·60	1·205	2·165	3·494	5·064
0·70	1·278	2·189	3·495	5·087
0·80	1·331	2·226	3·495	5·110
	$\mathrm{Bi}_q > 1{\cdot}0$			
1·5	1·587	2·409	3·507	5·227
2·0	1·663	2·542	3·512	5·290
3·0	1·735	2·748	3·525	5·382
4·0	1·767	2·890	3·535	5·440
5·0	1·784	2·991	3·546	5·471
6·0	1·795	3·062	3·557	5·491
7·0	1·803	3·113	3·573	5·506
8·0	1·808	3·150	3·588	5·515
9·0	1·812	3·177	3·601	5·524
10	1·815	3·198	3·614	5·530
15	1·825	3·264	3·650	5·544
20	1·831	3·297	3·676	5·550
30	1·835	3·329	3·708	5·556
40	1·837	3·346	3·721	5·558
50	1·839	3·354	3·732	4·560

$$N = P_2\left[\nu_1\,\mu\cos\nu_1\,\mu + \frac{1-\varepsilon}{\varepsilon}\,\mathrm{Lu}\,\mathrm{Bi}_m(1-\nu_1^2)\sin\nu_1\,\mu\right] -$$

$$- P_1\left[\nu_2\,\mu\cos\nu_2\,\mu + \frac{1-\varepsilon}{\varepsilon}\,\mathrm{Lu}\,\mathrm{Bi}_m(1-\nu_2^2)\sin\nu_2\,\mu\right].$$

The roots μ_n of the characteristic equation are determined by a graphical method analogous to that described above. The first four roots are shown in Table 13. The initial values of Fo for approximate calculation of heat and mass transfer in the sphere are given in Table 6. A comparison of these

data with the corresponding data for an infinite plate shows that, in the problem of the sphere, the initial values of Fo are considerably less than in the problem of the plate.

Table 14 gives the coefficient of heat and mass transfer for the sphere.

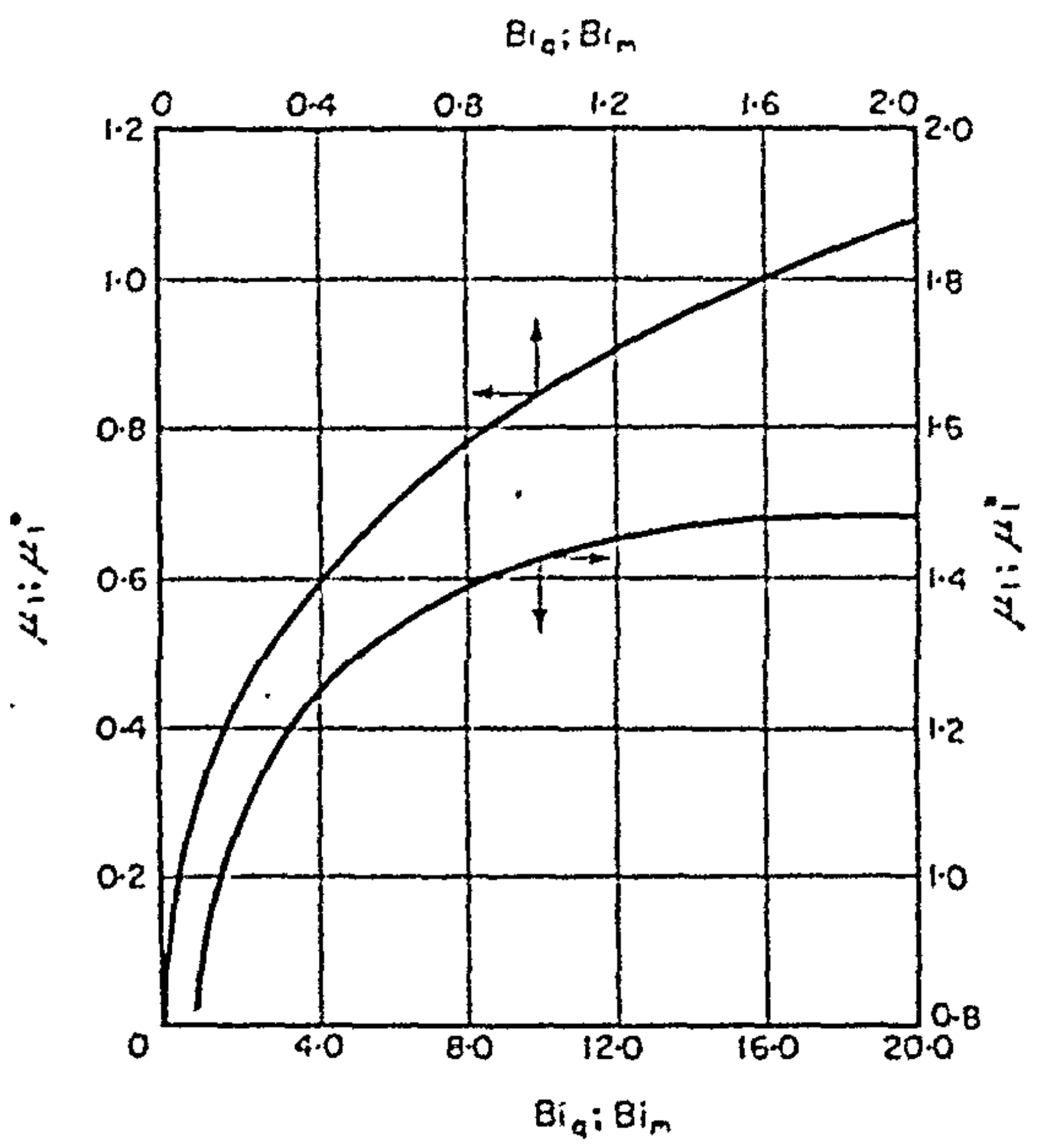


FIG. 17. Relation between the first roots of the characteristic equations $\cot \mu = \mu/\mathrm{Bi}_q$ and $\cot \mu^* = \mu^*/\mathrm{Bi}_m$ and the criterin of Biot for a plate

The mean values of the transfer potentials for heat and mass in the case of the sphere can be obtained from the relations

$$\bar{t}(\tau) = \frac{3}{R^3} \int_0^R r^2 t(r, \tau)\, dr\,, \quad \bar{\Theta}(\tau) = \frac{3}{R^3} \int_0^R r^2 \Theta(r, \tau)\, dr\,.$$

For the mean non-dimensional transfer potentials we have:

$$\bar{T} = \frac{\bar{t}(\tau) - t_0}{t - t_0} = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 \mathrm{Fo})\,, \qquad (1.80)$$

$$\bar{\Theta} = \frac{\theta_0 - \bar{\theta}(\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}). \quad (1.81)$$

Here

$$D_n = D_{n1} - D_{n2}, \quad D_n^* = D_{n2}^* - D_{n1}^*, \quad (1.82)$$

$$D_{n1} = \frac{3(\sin \nu_2 \mu_n - \nu_2 \mu_n \cos \nu_2 \mu_n)}{\nu_2^2 \mu_n^2} C_{n1},$$

$$D_{n2} = \frac{3(\sin \nu_1 \mu_n - \nu_1 \mu_n \cos \nu_1 \mu_n)}{\nu_1^2 \mu_n^2} C_{n2},$$

$$D_{n1}^* = \frac{3(\sin \nu_2 \mu_n - \nu_2 \mu_n \cos \nu_2 \mu_n)}{\nu_2^2 \mu_n^2} (1 - \nu_2^2) C_{n1}^*,$$

$$D_{n2}^* = \frac{3(\sin \nu_1 \mu_n - \nu_1 \mu_n \cos \nu_1 \mu_n)}{\nu_1^2 \mu_n^2} (1 - \nu_1^2) C_{n2}^*.$$

The coefficients D_n and D_n^* calculated from equation (1.82) for different values of Bi_q are collected in Table 14.

TABLE 14. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER TO Bi_q.

Bi_q	C_{n1}		C_{n2}		C_{n1}^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5	0·1588	0·009269	—0·1211	—0·2415	0·2647	0·01545
10	0·1554	0·01978	—0·03765	—0·2384	0·2590	0·03297
15	0·1539	0·02190	—0·01254	—0·2022	0·2565	0·03650
20	0·1530	0·02288	0·00001812	—0·1772	0·2550	0·03813

Bi_q	C_{n2}^*		D_n		D_n^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5	—0·2018	—0·4025	0·3272	0·2861	0·3984	—0·09914
10	—0·06275	—0·3973	0·2181	0·2583	0·4138	—0·1035
15	—0·02090	—0·3370	0·1845	0·2107	0·4173	—0·09213
20	0·00003020	—0·2953	0·1669	0·1800	0·4178	—0·08417

The non-dimensional rate of change of the potentials can be obtained by differentiating equations (1.77) and (1.78)

for local values and (1.80) and (1.81) for mean values relative to the non-dimensional time.

We have:

$$\frac{dT}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \left(C_{n1} \frac{\sin \nu_2 \mu_n r/R}{r/R} - \right.$$

$$\left. - C_n{}^2 \frac{\sin \nu_1 \mu_n r/R}{r/R} \right) \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.83)$$

$$\frac{d\bar{T}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.84)$$

$$\frac{d\,\Theta}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \left[C_{n2}^{*}(1-\nu_1^2) \frac{\sin \nu_1 \mu_n r/R}{r/R} - \right.$$

$$\left. - C_{n1}^{*}(1-\nu_2^2) \frac{\sin \nu_2 \mu_n r/R}{r/R} \right] \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.85)$$

$$\frac{d\,\overline{\Theta}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^{*} \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right). \qquad (1.86)$$

The non-dimensional quantities ν_1 and ν_2 appear in the solutions given, along with the known heat and mass exchange criteria of similarity. The role of ν_j, unlike that of the other criteria, is bound up not with a physical process but with a mathematical one. In the general case (see equation (1.5)) ν_j depends on the criteria of Fedorov and Lu:

$$\nu_j = \nu_j\,(\mathrm{Fe}, \mathrm{Lu}).$$

Table 15 gives the values of ν_j in intervals of the change of Lu from 0 to 100, and of the Fedorov criterion from 0 to 1·0. It will be seen from these tables that ν_1 does not change significantly with Lu and Fe. With increase of each of these criteria it diminishes. For $\mathrm{Lu} = 0$, ν_1 is equal to unity, and for $\mathrm{Lu} > 1{\cdot}0$ it diminishes and tends to zero. With increase of the criterion Fe, ν_2 increases very little, but with increasing Lu it diminishes approximately according to a logarithmic curve.

TABLE 15. VALUES OF $\nu_j = \nu_j$ (Lu, Fe) $j = 1{\cdot}2$.

Lu \ Fe	ν_j	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	ν_1	1	1	1	1	1	1	1	1	1	1	1
	ν_2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
	ν_1	1	0·9995	0·9990	0·9985	0·9980	0·9975	0·9970	0·9965	0·9960	0·9955	0·9950
0·01	ν_2	10·0000	10·0050	10·0100	10·0150	10·0201	10·0251	10·0301	10·0352	10·0403	10·0453	10·0503
	ν_1	1	0·9990	0·9980	0·9969	0·9960	0·9949	0·9939	0·9929	0·9919	0·9909	0·9899
0·02	ν_2	7·0711	7·0782	7·0854	7·0926	7·0998	7·1070	7·1141	7·1213	7·1285	7·1356	7·1428
	ν_1	1	0·9979	0·9959	0·9938	0·9918	0·9837	0·9877	0·9858	0·9838	0·9838	0·9798
0·04	ν_2	5·0000	5·0104	5·0208	5·0311	5·0414	5·0517	5·0620	5·0722	5·0825	0·0927	5·1029
	ν_1	1	0·9968	0·9937	0·9906	0·9875	0·9845	0·9815	0·9784	0·9754	0·9725	0·9696
0·06	ν_2	4·0825	4·0955	4·1084	4·1213	4·1341	4·1469	4·1597	4·1724	4·1850	4·1977	4·2102
	ν_1	1	0·9957	0·9914	0·9873	0·9831	0·9791	0·9750	0·9710	0·9671	0·9632	0·9595
0·08	ν_2	3·5355	3·5508	3·5660	3·5812	3·5963	3·6113	3·6262	3·6410	3·6557	3·6704	2·6850
	ν_1	1	0·9945	0·9891	0·9838	0·9786	0·9735	0·9684	0·9635	0·9587	0·9539	0·9492
0·10	ν_2	3·1623	3·1798	3·1971	3·2143	3·2314	3·2484	3·2652	3·2820	3·2986	3·3151	3·3315
	ν_1	1	0·9878	0·9762	0·9651	0·9544	0·9440	0·9344	0·9250	0·9158	0·9071	0·8986
0·20	ν_2	2·2361	2·2636	2·2907	2·3170	2·3428	2·3681	2·3931	2·4175	2·4415	2·4652	2·4884
	ν_1	1	0·9796	0·9609	0·9437	0·9278	0·9129	0·8989	0·8858	0·8733	0·8615	0·8503
0·30	ν_2	1·8257	1·864	1·9000	1·9346	1·968	2·0000	2·031	2·0612	2·0906	2·1192	2·1472
	ν_1	1	0·9694	0·9431	0·9200	0·8993	0·8803	0·8632	0·8473	0·8325	0·8187	0·8057
0·40	ν_2	1·5811	1·6310	1·6764	1·7185	1·7582	1·7953	1·8310	1·8660	1·8992	1·9312	1·9623
	ν_1	1	0·9415	0·9004	0·8677	0·8402	0·8167	0·7955	0·7766	0·7594	0·7437	0·7292
0·60	ν_2	1·2910	1·3712	1·4339	1·4879	1·5365	1·5811	1·6229	1·6624	1·7000	1·7360	1·7706
	ν_1	1	0·9020	0·8511	0·8137	0·7835	0·7580	0·7358	0·7162	0·6985	0·6824	0·6676
0·80	ν_2	1·1180	1·2395	1·3136	1·3740	1·4269	1·4750	1·5193	1·5611	1·6007	1·6384	1·6746
	ν_1	1	0·8543	0·8011	0·7630	0·7326	0·7071	0·6850	0·6657	0·6482	0·6324	0·6180
1·00	ν_2	1·0000	1·1705	1·2483	1·3107	1·3650	1·4142	1·4597	1·5023	0·5427	1·5811	1·6181

TABLE 15. *Contd.*

Lu \ Fe	v_f	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	v_1 v_2	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞	1 ∞
2·00	v_1	1	0·6526	0·6150	0·5858	0·5617	0·5409	0·5233	0·5073	0·4931	0·4801	0·4682
	v_2	0·7071	1·0836	1·1496	1·2070	1·2587	1·3067	1·3513	1·3938	1·4342	1·4730	1·5102
4·00	v_1	1	0·4706	0·4472	0·4276	0·4108	0·3960	0·3831	0·3714	0·3609	0·3512	0·3423
	v_2	0·5000	1·0623	1·1180	1·1693	1·2171	1·2622	1·3050	1·3461	1·3856	1·4236	1·4604
6·00	v_1	1	0·3863	0·3680	0·3523	0·3389	0·3267	0·3162	0·3066	0·2979	0·2900	0·2826
	v_2	0·4083	1·0571	1·1097	1·1587	1·2049	1·2487	1·2910	1·3314	1·3704	1·4080	1·4446
8·00	v_1	1	0·3352	0·3198	0·3062	0·2949	0·2846	0·2753	0·2670	0·2594	0·2525	0·2461
	v_2	0·3536	1·0548	1·1058	1·1544	1·1992	1·2524	1·2842	1·3243	1·3629	1·4004	1·4368
10·0	v_1	1	0·3002	0·8865	0·2748	0·2645	0·2556	0·2470	0·2396	0·2328	0·2265	0·2208
	v_2	0·3162	1·0535	1·1036	1·1509	1·1958	1·2388	1·2802	1·3201	1·3586	1·3960	1·4322
20·0	v_1	1	0·2127	0·2034	0·1952	0·1880	0·1817	0·1758	0·1704	0·1657	0·1612	0·1572
	v_2	0·2236	1·0510	1·0994	1·1454	1·1894	1·2318	1·2724	1·3118	1·3500	1·3870	1·4231
30·0	v_1	1	0·1737	0·1661	0·1595	0·1538	0·1488	0·1437	0·1394	0·1355	0·1319	0·1285
	v_2	0·1825	1·0503	1·0981	1·1436	1·1837	1·2293	1·2629	1·3091	1·3472	1·3841	1·4201
40·0	v_1	1	0·1506	0·1439	0·1384	0·1332	0·1285	0·1247	0·1208	0·1174	0·1145	0·1114
	v_2	0·1581	1·0499	1·0974	1·1427	1·1863	1·2282	1·2686	1·3078	1·3458	1·3827	1·4187
50·0	v_1	1	0·1347	0·1288	0·1239	0·1194	0·1153	0·1116	0·1082	0·1051	0·1022	0·0998
	v_2	0·1414	1·0497	1·0970	1·1422	1·1857	1·2275	1·2679	1·3070	1·3450	1·3819	1·4177
60·0	v_1	1	0·1230	0·1179	0·1131	0·1091	0·1054	0·1020	0·0990	0·0962	0·0935	0·0911
	v_2	0·1292	1·0495	1·0967	1·1419	1·1852	1·2270	1·2674	1·3065	1·3441	1·3812	1·4172
80·0	v_1	1	0·1066	0·1020	0·0980	0·0943	0·0909	0·0883	0·0854	0·0831	0·0810	0·0791
	v_2	0·1118	1·0493	1·0964	1·1415	1·1847	1·2265	1·2668	1·3058	1·3437	1·3805	1·4161
100	v_1	1	0·0954	0·0911	0·0875	0·0843	0·0817	0·0791	0·0765	0·0746	0·0725	0·0707
	v_2	0·1000	1·0492	1·0962	1·1412	1·1845	1·2261	2·2663	1·3054	1·3433	1·3801	0·4160

(c) *Infinite plate. Parabolic initial conditions*

$$\left[\Gamma = 0,\ \xi = x,\ f(x) = t(x, 0) = t_c + \left(\frac{x}{R}\right)^2 (t_w - t_c)\,\lambda\right]$$

$$\varphi(x) = \Theta(x, 0) = \theta_c - \left(\frac{x}{R}\right)^2 (\theta_c - \theta_n)$$

For the solution of this case we shall use the operational method in the form of a Laplace transformation. Applying it to equations (I–II) we obtain:

$$a_q t''_L(x, s) - s t_L(x, s) +$$

$$+ \frac{\varepsilon\varrho}{c} s\,\theta_L(x, s) - \frac{\varepsilon\varrho}{c}\theta(x, 0) - t(x, 0) = 0\,. \qquad (1.87)$$

$$a_m \theta''_L(x, s) - s\,\theta(x, s) + a_m \delta\, t''_L(x, s) + \theta(x, 0) = 0\,. \qquad (1.88)$$

We find $\theta_L(x, s)$, $\theta''_L(x, s)$ from equation (1.87) and substitute these expressions in (1.88). After transformation we obtain the non-homogeneous ordinary differential equation (1.89)

$$t^{IV}_L(x, s) - \frac{s}{a_q} \Lambda\, t''_L(x, s) - \frac{s^2}{a_q a_m} t_L(x, s) +$$

$$+ \frac{1}{2} w x^2 + \varkappa = 0\,, \qquad (1.89)$$

where

$$\Lambda = 1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\,,$$

$$w = 2\,\frac{t_u - t_n}{a_q R^2}\,,$$

$$\varkappa = \frac{s}{a_q a_m} t_u + w - 2\,\frac{\varepsilon\varrho}{c}\,\frac{\theta_u - \theta_n}{a_c R^2}\,.$$

The general integral of equation (1.89) is found by the double use of the Laplace transformation with respect to the co-ordinate x. We have

$$t_L(x, s) = \frac{t_u - (t_u - t_n)\left(\frac{x^2}{R^2}\right)}{s} + \frac{2\,a_m \omega}{s^2} + C_1 e^{\sqrt{\left(\frac{s}{a_q}\right)} r_1 x} +$$

$$+ C_2 e^{\sqrt{\left(\frac{s}{a_q}\right)} r_2 x} + C_3 e^{-\sqrt{\left(\frac{s}{a_q}\right)} r_1 x} + C_4 e^{-\sqrt{\left(\frac{s}{a_q}\right)} r_2 x}\,, \qquad (1.90)$$

where

$$\omega = \frac{t_u - t_n}{R^2} - \frac{\varepsilon\varrho}{c}\,\frac{\theta_u - \theta_n}{R^2} - \frac{t_u - t_n}{R^2}\,\Lambda\,, \qquad (1.91)$$

r_j is defined by equation (1.5).
C_k ($k = 1, 2, 3, 4$) are constants.

A solution for the transform of the mass potential can be obtained by substituting $t_L(x, s)$ and $t_L''(x, s)$ in equation (1.87), i.e.

$$\theta_L(x, s) = \frac{\theta_u - (v_u - \theta_n)\,\frac{x^2}{R^2}}{s} + \frac{a_q c}{\varepsilon\varrho}\,\frac{2\left(\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right)}{s^2} +$$

$$+ \frac{c}{\varepsilon\varrho}\Big[C_1(1 - r_1^2)\,e^{\sqrt{\left(\frac{s}{a_q}\right)}r_1 x} + C_2(1 - r_2^2)\,e^{\sqrt{\left(\frac{s}{a_q}\right)}r_2 x} +$$

$$+ C_3(1 - r_1^2)\,e^{-\sqrt{\left(\frac{s}{a_q}\right)}r_1 x} + C_4(1 - r_2^2)\,e^{-\sqrt{\left(\frac{s}{a_q}\right)}r_2 x}\Big]. \qquad (1.92)$$

From the condition of symmetry (V) it follows that $C_1 = C_3$ and $C_2 = C_4$, whence (1.90) and (1.92) can be rewritten:

$$t_L(x, s) = \frac{t_u - (t_u - t_n)\left(\frac{r^2}{R^2}\right)}{s} + \frac{2\,a_m\omega}{s^2} +$$

$$+ B_1 \cosh\sqrt{\left(\frac{s}{a_q}\right)}r_1 x + B_2 \cosh\sqrt{\left(\frac{s}{a_q}\right)}r_2 x\,, \qquad (1.93)$$

$$\theta_L(x, s) = \frac{\theta_u - (\theta_u - \theta_n)\,\frac{x^2}{R^2}}{s} + \frac{2\,\frac{a_q c}{\varepsilon\varrho}\left(\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right)}{s^2} +$$

$$+ B_1\,\frac{c}{\varepsilon\varrho}(1 - r_1^2)\cosh\sqrt{\left(\frac{s}{a_q}\right)}r_1 x +$$

$$+ B_2\,\frac{c}{\varepsilon\varrho}(1 - r_2^2)\cosh\sqrt{\left(\frac{s}{a_q}\right)}r_2 x\,. \qquad (1.94)$$

where B_1 and B_2 are new constants with respect to x.

Adjusting the solutions (1.93) and (1.94) to the boundary conditions (1.8) and (1.9) we have for B_j the system of equations (1.95)

$$\left.\begin{aligned} B_1 Q_1 + B_2 Q_2 - \frac{\Omega_1}{s} - \frac{\Omega_2}{s^2} = 0 \\ B_1 P_1 + B_2 P_2 - \frac{\Omega_3}{s} - \frac{\Omega_4}{s^2} = 0 \end{aligned}\right\}, \tag{1.95}$$

where

$$\Omega_1 = \frac{2}{\mathrm{Bi}_q}(t_u - t_n) + (t_c - t_n) - \varepsilon K_1 \frac{\varrho}{c}(\theta_n - \theta_p), \tag{1.96}$$

$$\Omega_2 = -2 a_q \left[K_1\left(\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right) + u \mathrm{Lu}\right], \tag{1.97}$$

$$\Omega_3 = 2 \frac{\mathrm{Fe}}{\mathrm{Bi}_m}\left[\frac{1}{\delta}(\theta_u - \theta_n) + (t_u - t_n) - \frac{\mathrm{Bi}_m}{2\delta}(\theta_n - \theta_p)\right], \tag{1.98}$$

$$\Omega_4 = -2 a_q \left(\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right), \tag{1.99}$$

and Q_j and P_j are determined respectively by equations (1.13) and (1.14). The solution of (1.95) gives

$$B_1 = -\left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_2 - Q_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right],$$

$$B_2 = \frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)}.$$

Substituting B_1 and B_2 in equations (1.93) and (1.94) we have finally for the distribution of transfer potentials in the representations the following general solutions:

$$t_L(x, s) = \frac{t_u - (t_u - t_n)\frac{x^2}{R^2}}{s} + \frac{2 a_m \omega}{s^2} +$$

$$+ \left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x - \right.$$

$$\left. - \left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_2 - \Omega_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x \right\}, \tag{1.100}$$

$$\theta_L(x, s) = \frac{\theta_u - (\theta_u - \theta_n)\frac{x^2}{R^2}}{s} + \frac{2\frac{a_q c}{\varepsilon \varrho}\left(\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right)}{s^2} +$$

$$+\left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \right.\right.$$

$$\left. + \frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \frac{c}{\varepsilon Q} (1 - \nu_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x -$$

$$- \left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \right.$$

$$\left. \left. + \frac{\Omega_2 P_2 - \Omega_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \frac{c}{\varepsilon Q} (1 - \nu_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x \right\}. \quad (1.101)$$

In these equations we have two groups of terms for which we have to obtain the originals; these are:

$$\frac{\Omega_1 P_j - \Omega_3 Q_j}{s(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} - \nu x,$$

and

$$\frac{\Omega_2 P_j - \Omega_4 Q_j}{s^2(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} - \nu x.$$

Here $\nu = \nu_{j-(-1)^j}$ $(j = 1, 2)$.

Consider a transform of the first type:

$$L^{-1}\left[\frac{(\Omega_1 P_j - \Omega_3 Q_j) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu x}{s(Q_2 P_1 - Q_1 P_2)}\right]$$

$$= L^{-1}\left[\frac{\Phi_{01}(s)}{\Psi_{01}(s)}\right]. \quad (1.102)$$

Since the conditions of the expansion theorem are fulfulled and the roots of the denominator $\Psi_{01}(s)$ are

$s = 0$ (null root) and

$s = s_n$, where $s_n = -\frac{a_q \mu_n^2}{R^2}$,

the expansion theorem may be written in the following form:

$$L^{-1}\left[\frac{\Phi_{01}(s)}{\Psi_{01}(s)}\right] = \frac{\Phi_{01}(0)}{\Psi'_{01}(0)} + \sum_{n=1}^{\infty} \frac{\Phi_{01}(s_n)}{\Psi'_{01}(s_n)} \exp(s_n \tau). \quad (1.102')$$

For $\Phi_{01}(0)$, $\Psi'_{01}(0)$, $\Phi_{01}(s_n)$, $\Psi'_{01}(s_n)$ calculation gives:

$$\Phi_{01}(0) = (\Omega_1 - K_1 \Omega_3)(1 - \nu_j^2) - \Omega_3,$$

$$\Psi'_{01}(0) = \nu_2^2 - \nu_1^2,$$

$$\Phi_{01}(s_n) = (\Omega_1 P_{nj} - \Omega_3 Q_{nj}) \cos \mu_n \underline{\nu} \frac{x}{R}.$$

$\psi'_{01}(s_n) = \psi_n \mu_n/2$, where μ_n, Q_{nj}, P_{nj} and ψ_n are described respectively by equations (1.20) – (1.23).

Let us substitute the expressions thus obtained in (1.102′), and hence in (1.102). We shall thus obtain the inversion of the first type of transform:

$$L^{-1}\left[\frac{(\Omega_1 P_j - \Omega_3 Q_j) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \underline{\nu}\, x}{s(Q_2 P_1 - Q_1 P_2)}\right]$$

$$= \frac{(\Omega_1 - K_1 \Omega_3)(1 - \nu_j^2) - \Omega_3}{\nu_2^2 - \nu_1^2} +$$

$$+ \sum_{n=1}^{\infty} 2 \frac{(\Omega_1 P_{nj} - \Omega_3 Q_{nj}) \cos \underline{\nu} \mu_n \frac{x}{R}}{\mu_n \psi_n} \exp(-\mu_n^2 \mathrm{Fo}). \quad (1.103)$$

Let us now consider the inversion of the second type of *transform*:

$$L^{-1}\left[\frac{(\Omega_2 P_j - \Omega_4 Q_j) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \underline{\nu}\, x}{s^2(Q_2 P_1 - Q_1 P_2)}\right]$$

$$= L^{-1}\left[\frac{\Phi_{02}(s)}{\psi_{02}(s)}\right] = L^{-1}\left[\frac{\Phi_{02}(s)}{s^2 \varphi(s)}\right].$$

The zero term with a double zero root constitutes an exception to the case considered above. To obtain it we will make use of the expansion theorem for multiple roots

$$L^{-1}\left[\frac{\Phi_{02}(0)}{\psi_{02}(0)}\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{\Phi_{02}(s)\, s^2}{\psi_{02}(s)} e^{s\tau}\right)\right]$$

$$= \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{\Phi_{02}(s)}{\varphi(s)} e^{s\tau}\right)\right]$$

$$= \lim_{s \to 0}\left[\frac{\tau\, e^{s\tau} \Phi_{02}(s)}{\varphi(s)} + \frac{e^{s\tau} \Phi'_{02}(s)}{\varphi(s)} - \frac{e^{s\tau} \Phi_{02}(s) \varphi'(s)}{[\varphi(s)]^2}\right].$$

The terms in the theorem have the following values:

$$\varphi(0) = \nu_2^2 - \nu_1^2,$$

$$\varphi'(0) = \frac{R^2}{2\,a_q}\left(1 + \mathrm{Fe} + \frac{2}{\mathrm{Bi}_q} + \frac{1}{\mathrm{Lu}} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}\right)(\nu_2^2 - \nu_1^2)\,,$$

$$\Phi_{02}(0) = (\Omega_2 - K_1\,\Omega_4)\,(1 - \nu_j^2) - \Omega_4\,,$$

$$\Phi'_{02}(0) = [(\Omega_2 - K_1\,\Omega_4)\,(1 - \nu_j^2) - \Omega_4]\,\frac{\nu_j^2\,R^2 + \underline{\nu}^2\,x^2}{2\,a_q} +$$

$$+ \left[\Omega_2\,\frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} - \Omega_4\,\frac{1}{\mathrm{Bi}_q}\right]\frac{\nu_j^2\,R^2}{a_q}\,,$$

$$\psi'_{02}(s_n) = -\frac{a_q}{2\,R^2}\,\mu_n^3\,\psi_n\,,$$

$$\Phi_{02}(s_n) = (\Omega_2\,P_{nj} - \Omega_4\,Q_{nj})\cos\underline{\nu}\mu_n\,x/R.$$

Hence the inversion of the transform of the second type may be written:

$$L^{-1}\left[\frac{(\Omega_2\,P_j - \Omega_4\,Q_j)\cosh\sqrt{\left(\frac{s}{a_q}\right)}\,\underline{\nu}\,x}{s^2(Q_2\,P_1 - Q_1\,P_2)}\right]$$

$$= \tau\,\frac{(\Omega_2 - K_1\,\Omega_4)\,(1 - \nu_j^2) - \Omega_4}{\nu_2^2 - \nu_1^2} +$$

$$+ \frac{[(\Omega_2 - K_1\,\Omega_4)\,(1 - \nu_j^2) - \Omega_4]\,\frac{\nu_j^2\,R^2 + \underline{\nu}^2\,x^2}{2\,a_q}}{\nu_2^2 - \nu_1^2} +$$

$$+ \frac{\left[\Omega_2\,\frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} - \Omega_4\,\frac{1}{\mathrm{Bi}_m}\right]\frac{\nu_j^2\,R^2}{a_q}}{\nu_2^2 - \nu_1^2} -$$

$$- \frac{\frac{R^2}{2\,a_q}\left(1 + \mathrm{Fe} + \frac{2}{\mathrm{Bi}_q} + \frac{1}{\mathrm{Lu}} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}\right)}{\nu_2^2 - \nu_1^2} \times$$

$$\times [(\Omega_2 - K_1\,\Omega_4)\,(1 - \nu_j^2) - \Omega_4] -$$

$$- \sum_{n=1}^{\infty} \frac{2\,R^2}{a_q}\,\frac{(\Omega_2\,P_{nj} - \Omega_4\,Q_{nj})\cos\underline{\nu}\,\mu_n\,\frac{x}{R}}{\mu_n^2\,\varphi_n} \times$$

$$\times \exp(-\mu_n^2\,\mathrm{Fo})\,. \qquad (1.104)$$

The original for the fields of transfer potential will be obtained by applying the expansion theorem of equations

(1.100) and (1.101); here we must take into account the solutions found for the inversions of the transforms (1.103) and (1.104). Inserting the values of A, ω, Ω_k, ($k = 1, 2, 3, 4$) by use of (1.91) and (1.96)–(1.99), simplifying the solutions and expressing in terms of the criteria of similarity, we can finally write:

$$T_1 = \frac{t(x, \tau) - t_n}{t_c - t_n} = 1 - \sum_{n=1}^{\infty} (C_{n2} \cos \nu_1 \mu_n x/R - C_{n1} \cos \nu_2 \mu_n x/R) \exp(-\mu_n^2 \mathrm{Fo}), \quad (1.105)$$

$$\Theta_1 = \frac{\theta_n - \theta(x, \tau)}{\theta_n - \theta_p} = 1 - \sum_{n=1}^{\infty} [C_{n1}^*(1 - \nu_2^2) \cos \nu_2 \mu_n x/R - C_{n2}^*(1 - \nu_1^2) \cos \nu_1 \mu_n x/R] \exp(-\mu_n^2 \mathrm{Fo}), \quad (1.106)$$

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \varphi_n},$$

$$C_{nj}^* = \frac{C_{nj}}{\varepsilon \mathrm{Ko}}, \quad (1.107)$$

$$G_1 = 1 - 2W \left(\frac{1}{\mathrm{Bi}_q} - \frac{1}{\mu_n^2}\right) - \varepsilon \mathrm{Ko} K_1 - 2 \varepsilon \mathrm{Ko} K_2 \frac{(K_1 + 1) \mathrm{Lu}}{\mu_n^2}, \quad (1.108)$$

$$G_2 = 2 \varepsilon \mathrm{Ko} K_2 \left(\frac{1}{\mathrm{Bi}_m} - \frac{\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon \mathrm{Ko}. \quad (1.109)$$

The criteria Lu, Bi_q, Bi_m, Ko, Pn, ε, K_1 and Fe in parabolic problems preserve their old form if we replace t_0 and θ_0 by t_n and θ_n.

In view of the identity of the characteristic equation in the parabolic problem with that for a plate under constant initial conditions, the values of μ_n given in Tables 1–4 are the roots for this problem also. A comparison of the initial values of Fo for simplified calculations under parabolic initial conditions (Table 6) with the analogous values for constant ini-

tial conditions shows no material differences. From this we may conclude that the initial conditions have little influence on the simplicity of the solutions of the system of equations for heat and mass transfer. The coefficients of heat and mass transfer calculated from equations (1.107) for a plate, as functions of the parametric criteria of non-uniformity of the initial distributions of W and V, are collected in Tables 16 and 17.

The mean values of the non-dimensional temperature and potential of mass transfer in the parabolic problem can be written:

$$\bar{T}_1 = \frac{\bar{t}(\tau) - t_n}{t_c - t_n} = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 \mathrm{Fo}), \quad (1.110)$$

$$\bar{\Theta} = \frac{\theta_n - \bar{\theta}(\tau)}{\theta_n - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}). \quad (1.111)$$

D_n and D_n^* preserve their old external form as in (1.38) and (1.39) with new values of C_{nj} and C_{nj}^*.

The non-dimensional rate of heating and drying of a plate can be obtained, as usual, by the differentiation of equations (1.105) and (1.106), or (1.110) and (1.111) with respect to the non-dimensional time, i.e.

$$\frac{dT_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} (C_{n2} \cos r_1 \mu_n x/R -$$

$$- C_{n1} \cos r_2 \mu_n x/R) \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\frac{dT_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\frac{d\Theta_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} [C_{n1}^*(1 - r_2^2) \cos r_2 \mu_n x/R -$$

$$- C_{n2}^*(1 - r_1^2) \cos r_1 \mu_n x/R] \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\frac{d\bar{\Theta}_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^* \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}).$$

TABLE 16. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF (INFINITE PLATE WITH PARA-

W	C_{n_1}		C_{n2}		$C^*_{n_1}$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0·3918	—0·01300	0·06262	0·8372	—0·6638	—0·02167
0·2	—0·3982	—0·01585	0·06286	1·026	—0·6638	—0·02642
0·4	—0·4046	—0·01868	0·06310	1·214	—0·6745	—0·03114
0·6	—0·4110	—0·02154	0·06336	1·403	—0 6851	—0·03591
0·8	—0·4174	—0·02438	0·06360	1·591	—0·6958	—0·04064
1·0	—0·4238	—0·02722	0·06384	1·779	—0·7065	—0·04538

TABLE 17. RELATION OF THE COEFFICIENTS IN THE SOLUTION OF (INFINITE PLATE WITH PARA-

V	C_{n_1}		C_{n2}		$C^*_{n_1}$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0·3208	—0·02020	0·04926	1·315	—0·5348	—0·03367
0·2	—0·3624	—0·01944	0·05615	1·264	—0·6045	—0·03241
0·4	—0·4046	—0·01868	0·06310	1·214	—0·6745	—0·03114
0·6	—0·4466	—0·01794	0·07002	1·164	—0·7445	—0·02991
0·8	—0·4882	—0·01718	0·07692	1·113	—0·8138	—0·02864
1·0	—0·5300	—0·01641	0·08384	1·063	—0·8835	—0·02736

(d) *Sphere. Parabolic initial conditions*

$$\left[\Gamma = 2,\ \xi = r,\ f(r) = t_u + \right.$$

$$\left. + \left(\frac{r}{R}\right)^2 (t_n - t_u),\ \varphi(r) = \theta_u - \left(\frac{r}{R}\right)^2 (\theta_u - \theta_n)\right].$$

The general solution for the transfer potentials of heat and mass can be obtained in transform by the method set out in the previous problem. They will be:
for the heat potential

$$\vartheta_L(r, s) = \frac{r\left[t_u - (t_u - t_n)\frac{r^2}{R^2}\right]}{s} + \frac{6\, r a_m \omega}{s^2} +$$

$$+ C_1 e^{\sqrt{\left(\frac{s}{a_q}\right)} r_1 r} + C_2 e^{\sqrt{\left(\frac{s}{a_q}\right)} r_2} +$$

THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER
BOLIC INITIAL CONDITIONS).

$C^*_{n_1}$		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·1044	1·396	0·3274	0·5802	1·223	—0·1015
0·1048	1·710	0·3320	0·7111	1·243	—0·1243
0·1052	2·024	0·3366	0·8414	1·263	—0·1472
0·1056	2·339	0·4313	0·9724	1·283	—0·1701
0·1060	2·652	0·3459	1·103	1·303	—0·1929
0·1064	2·966	0·3505	1·233	1·324	—0·2157

THE SYSTEM OF EQUATIONS OF HEAT AND MASS TRANSFER
BOLIC INITIAL CONDITIONS).

C^*_2		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·08212	2·102	0·2662	0·9114	1·002	—0·1594
0·09362	2·107	0·3012	0·8761	1·133	—0·1532
0·1052	2·024	0·3366	0·8414	1·263	—0·1472
0·1167	1·940	0·3719	0·8068	1·394	—0·1411
0·1282	1·855	0·4069	0·7714	1·524	—0·1349
0·1398	1·772	0·4420	0·7368	1·655	—0·1288

$$+ C_3 e^{-\sqrt{\left(\frac{s}{a_t}\right)} r_1 r} + C_4 e^{-\sqrt{\left(\frac{s}{a_t}\right)} r_2 r}; \qquad (1.112)$$

for the mass transfer potential

$$u_L(r, s) = \frac{r\left[\theta_u - (\theta_u - \theta_n)\frac{r^2}{R^2}\right]}{s} +$$

$$+ \frac{a_q c}{\varepsilon \varrho} \frac{6r\left[\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right]}{s^2} +$$

$$+ \frac{c}{\varepsilon \varrho}\left[C_1(1 - r_1^2) e^{\sqrt{\left(\frac{s}{a_t}\right)} r_1 r} + C_2(1 - r_2^2) e^{\sqrt{\left(\frac{s}{a_t}\right)} r_1 r} +\right.$$

$$\left. + C_3(1 - r_1^2) e^{-\sqrt{\left(\frac{s}{a_t}\right)} r_1 r} + C_4(1 - r_2^2) e^{-\sqrt{\left(\frac{s}{a_t}\right)} r_2 r}\right], \qquad (1.113)$$

where C_k ($k = 1, 2, 3, 4$) are constant with respect to r, and v_j and ω are given by equations (1.5) and (1.91).

To simplify equations (1.112) and (1.113) we will make use of the conditions of symmetry. In the analysis of problems with constant initial conditions it was shown that for the sphere the symmetry conditions (V) could be represented in the transform by equations (1.51) and (1.52),

$$\vartheta_L(0, s) = 0, \; u_L(0, s) = 0 .$$

Application of these conditions to equations (1.112) and (1.113) gives

$$C_1 = -C_3, \; C_2 = -C_4 .$$

Hence the solutions for the transform of the modified temperature $\vartheta_L(r, s)$ and the transfer potential of mass $u_L(r, s)$ will be

$$\vartheta_L(r, s) = \frac{r\left[t_u - (t_u - t_n)\frac{r^2}{R^2}\right]}{s} + \frac{6\, r a_m \omega}{s^2} + \\ + B_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 r + B_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 r, \quad (1.114)$$

$$u_L(r, s) = \frac{r\left[\theta_u - (\theta_u - \theta_n)\frac{r^2}{R^2}\right]}{s} + \frac{6\, r \frac{a_q c}{\varepsilon \varrho}\left[\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right]}{s^2} + \\ + B_1 \frac{c}{\varepsilon \varrho}(1 - v_1^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 r + \\ + B_2 \frac{c}{\varepsilon \varrho}(1 - v_2^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 r. \quad (1.115)$$

B_1 and B_2 are new constants relative to the coordinate whose values are defined by the boundary conditions (III–IV), or in transformation by (1.57) and (1.58). Substituting (1.114) and (1.115) in the boundary conditions and solving the system of equations thus obtained for B_1 and B_2 we have

$$B_1 = -\left[\frac{\Omega_1 \tilde{P}_2 - \Omega_3 \tilde{Q}_2}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} + \frac{\Omega_2 \tilde{P}_2 - \Omega_4 \tilde{Q}_2}{s^2(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)}\right],$$

$$B_2 = \frac{\Omega_1 \tilde{P}_1 - \Omega_3 \tilde{Q}_1}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} + \frac{\Omega_2 \tilde{P}_1 - \Omega_4 \tilde{Q}_1}{s^2(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} \cdots$$

Here $\tilde{Q}_j$ and $\tilde{P}_j$ are defined by equations (1.60) and (1.61), Ω_1 and Ω_3 by equations (1.96) and (1.98), and

$$\Omega_2 = -6\,a_q\left[K_1\left(\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right) + \omega\,\mathrm{Lu}\right], \quad (1.116)$$

$$\Omega_4 = -6\,a_q\left[\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right]. \quad (1.117)$$

Hence the solution of the system in transform can be finally written:

$$\vartheta_L(r, s) = \frac{r\left[t_u - (t_u - t_n)\,\frac{r^2}{R^2}\right]}{s} + \frac{6\,r a_m \omega}{s^2} +$$

$$+\left\{\left[\frac{\Omega_1\tilde{P}_1 - \Omega_3\tilde{Q}_1}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} + \frac{\Omega_2\tilde{P}_1 - \Omega_4\tilde{Q}_1}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right]\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 r -\right.$$

$$\left. -\left[\frac{\Omega_1\tilde{P}_2 - \Omega_3\tilde{Q}_2}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} + \frac{\Omega_2\tilde{P}_2 - \Omega_4\tilde{Q}_2}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right]\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 r\right\}, \quad (1.118)$$

$$u_L(r, s) = \frac{r\left[\theta_u - (\theta_u - \theta_n)\,\frac{r^2}{R^2}\right]}{s} +$$

$$+\frac{c}{\varepsilon\varrho}\left\{\frac{6\,r a_q\left[\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right]}{s^2} +\right.$$

$$+\left[\frac{\Omega_1\tilde{P}_1 - \Omega_3\tilde{Q}_1}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} + \frac{\Omega_2\tilde{P}_1 - \Omega_4\tilde{Q}_1}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right]\times$$

$$\times\,(1 - \nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 r - \left[\frac{\Omega_1\tilde{P}_2 - \Omega_3\tilde{Q}_2}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} +\right.$$

$$\left.\left. + \frac{\Omega_2\tilde{P}_2 - \Omega_4\tilde{Q}_2}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right](1 - \nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 r\right\}. \quad (1.119)$$

We will now obtain the two fundamental inversions of the transforms required for the carrying out of the inverse trans-

formation of equations (1.118) and (1.119):

$$L^{-1}\left[\frac{(\Omega_1 \widetilde{P}_j - \Omega_3 \widetilde{Q}_j)\sinh \sqrt{\left(\frac{s}{a_q}\right)}\, r}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)}\right] = L^{-1}\left[\frac{\Phi_1(s)}{\psi_1(s)}\right]. \quad (1.120)$$

Since the numerator and denominator of equation (1.120) are not generalized polynomials with respect to s, the expansion theorem for finding the original can be applied only after a series of transformations: the numerator and the denominator for the zero root are expanded in series, and both are divided by $\sqrt{(s/a_q)}$. The derivative of $\psi_{01}(s)$ with respect to s is taken, and the limit of the expression is found as s tends to zero (see § 1 b.).

Thus the expansion theorem for the expression (1.120) can be written:

$$L^{-1}\left|\frac{\Phi_1(s)}{\psi_1(s)}\right| = \frac{\Phi_{01}(0)}{\psi_1'(0)} + \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi_1(s_n)} \exp(s_n \tau), \quad (1.121)$$

where the first component corresponds to the zero root of $\psi_1(s) = s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2) = 0$ and the second to the roots $s_n = -a_q \mu_n^2 / R^2$, the expression $\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2 = 0$; and μ_n is determined by the roots of (1.65).

Solution of (1.120) or (1.121) gives

$$L^{-1}\left[\frac{(\Omega_1 \widetilde{P}_j - \Omega_3 \widetilde{Q}_j)\sinh \sqrt{\left(\frac{s}{a_q}\right)}\, r}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)}\right] = \frac{(Q_1 - K_1 \Omega_3)(1 - \nu_j^2) - \Omega_3}{\nu_2^2 - \nu_j^2} r =$$

$$= \sum_{n=1}^{\infty} 2 \frac{\Omega_1 P_{nj} - \Omega_3 Q_{nj}}{\mu_n \psi_n} R \sin \nu \mu_n \frac{r}{R} \exp(-\mu_n^2 \mathrm{Fo}). \quad (1.122)$$

The inverse transform of terms of the second type will be obtained in the same way as in the case of the problem of the plate, taking into account the additional transformations necessary to reduce the polynomials to a

condition which satisfies the generalized theorem of expansion. We have:

$$L^{-1}\left[\frac{(\Omega_1\bar{P}_j-\Omega_4\bar{Q}_j)\sinh\sqrt{\left(\frac{s}{a_s}\right)}r}{s^2(\bar{Q}_2\bar{P}_1-\bar{Q}_1\bar{P}_2)}\right]=r\left[(\Omega_2-K_1\Omega_4)(1-r_j^2)-\Omega_4\right]\times$$

$$\times\frac{\left[\tau+\frac{r^2r^2+R^2r_j^2}{6a_s}-\left(1+\frac{1}{\mathrm{Lu}}+\mathrm{Fe}+\frac{2}{\mathrm{Bi}_s}+\frac{2}{\mathrm{Bi}_m}\frac{1}{\mathrm{Lu}}\right)\frac{R^2}{6a_s}\right]}{r_2^2-r_1^2}+$$

$$+\frac{\left[\Omega_2\frac{(1-r_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}-\Omega_4\frac{1}{\mathrm{Bi}_s}\right]\frac{2r_j^2R^2}{6a_s}}{r_2^2-r_1^2}r+$$

$$+\sum_{n=1}^{\infty}2\frac{R^2}{a_s}\frac{\Omega_2P_{nj}-\Omega_4Q_{nj}}{\mu_n^2\gamma_n}R\sin r\mu_n\frac{r}{R}\exp(-\mu_n^2\mathrm{Fo}). \tag{1.123}$$

The quantities γ_n, A_{nj}, B_{nj} are determined by equations (1.70) to (1.72).

Applying the expansion theorem to equations (1.118) and (1.119), and taking account of the inversions (1.122) and (1.123), after transformations and the insertion of the values of A, ω, Ω_k ($k=1, 2, 3, 4$) we have for the field of temperature and mass potential:

$$T_1=\frac{t(r,\tau)-t_n}{t_c-t_n}=1-\sum_{n=1}^{\infty}\left[C_{n1}\frac{\sin r_1\mu_n r/R}{r/R}-\right.$$

$$\left.-C_{n2}\frac{\sin r_1\mu_n r/R}{r/R}\right]\exp(-\mu_n^2\mathrm{Fo}), \tag{1.124}$$

$$\Theta_1=\frac{\vartheta_n-\vartheta(r,\tau)}{\vartheta_n-\vartheta_p}=1-\sum_{n=1}^{\infty}\left[C_{n2}^*(1-r_1^2)\frac{\sin r_1\mu_n r/R}{r/R}-\right.$$

$$\left.-C_{n1}^*(1-r_2^2)\frac{\sin r_2\mu_n r/R}{r/R}\right]\exp(-\mu_n^2\mathrm{Fo}), \tag{1.125}$$

$$C_{nj}=2\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\gamma_n},\quad C_{nj}^*=\frac{1}{\varepsilon\,\mathrm{Ko}}C_{nj}, \tag{1.126}$$

$$G_1 = 1 - 2\,W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) -$$

$$- \varepsilon\,\mathrm{Ko}\,K_1 - 6\,\varepsilon\,\mathrm{Ko}\,K_2\,\frac{(K_1+2)\,\mathrm{Lu}}{\mu_n^2}, \qquad (1.127)$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,. \qquad (1.128)$$

The expressions for the non-dimensional temperature and the non-dimensional transfer potential of mass in this case will be:

$$\overline{T}_1 = \frac{\bar{t}(\tau) - t_n}{t_c - t_n} = 1 - \sum_{n=1}^{\infty} D_n \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (1.129)$$

$$\Theta_1 = \frac{\theta_n - \bar{\theta}(\tau)}{\theta_n - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (1.130)$$

where D_n and D_n^* retain the form of equation (1.82) taking account of the fact that C_{nj} and C_{nj}^* are determined by equation (1.126).

The mean and local rates of heating and drying are easily obtained by differentiation of the corresponding expressions with respect to the non-dimensional time Fo.

§ 2. Non-stationary Fields of Transfer Potential of Heat and Mass. Temperature of the Medium — a Linear Function of the Time

The actual processes in most cases take place in a medium with varying temperature. The body, which is in contact with a carrier of heat, warms up, and the heat carrier, giving up heat to the body, cools down. The mathematical picture of the process of internal heat and mass exchange has to reflect in its boundary conditions the changing temperature of the medium. The solution of the system of differential equations (I–VI) with changing temperature of the medium is primarily interesting for the description of drying, low temperature carbonization, humidification and gasification, not to mention the need for such solutions in order to work out

methods for determining the coefficients of heat and mass transfer. Below we consider problems where the temperature of the medium is a linear function of the time. Problems where the temperature of the medium changes exponentially with the time are given in the subsequent paragraph.

(a) *Infinite plate. Constant initial conditions* ($\Gamma = 0$, $\xi = x$, $t(x, 0) = t_0 = \text{const.}$, $\theta(x, 0) = \theta_0 = \text{const.}$, $t_c = t_c^0 - b\tau$).

The general solution of equations (I–VI) for the transfer potentials of heat and mass, taking account of the symmetry conditions (V) can be written in the following form:

$$t_L(x, s) - \frac{t_0}{s} = B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x + B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x, \quad (2.1)$$

$$\theta_L(x, s) - \frac{\theta_0}{s} = \frac{c}{\varepsilon\varrho} (1 - v_1^2) B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x +$$

$$+ \frac{c}{\varepsilon\varrho} (1 - v_2^2) B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x, \quad (2.2)$$

where

$$v_j^2 = \frac{1}{2} \left\{ \left(1 + \text{Fe} + \frac{1}{\text{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left[\left(1 + \text{Fe} + \frac{1}{\text{Lu}}\right)^2 - \frac{4}{\text{Lu}}\right]} \right\}, \ (j = 1, 2). \quad (2.3)$$

B_j are constants with respect to the coordinates, their values being determined from the boundary conditions (III–IV). The boundary conditions after the application of the Laplace transformation will be:

$$-\lambda t'_L(R, s) + \alpha \left[\frac{t_c^0}{s} - \frac{b}{s^2} - t_L(R, s)\right] -$$

$$- (1 - \varepsilon) \varrho\beta \left[\theta_L(R, s) - \frac{\theta_0}{s}\right] = 0, \quad (2.4)$$

$$\lambda_m \theta'_L(R, s) + \lambda_m \delta t'_L(R, s) + \\ + \beta\left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0 . \qquad (2.5)$$

Substituting in (2.4) and (2.5) $t_L(R, s)$, $t'_L(R, s)$, $\theta_L(R, s)$ and $\theta'_L(R, s)$ we solve the system of algebraical equations thus formed for B_1 and B_2. It follows from calculation that

$$B_1 = -\left\{\frac{(t_c^0 - t_0) P_2 + \frac{\varepsilon\varrho}{c}(\theta_0 - \theta_p)(Q_2 - K_1 P_2)}{s(Q_2 P_1 - Q_1 P_2)} - \right. \\ \left. - \frac{bP_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right\},$$

$$B_2 = \frac{(t_c^0 - t_0) P_1 + \frac{\varepsilon\varrho}{c}(\theta_0 - \theta_p)(Q_1 - K_1 P_1)}{s(Q_2 P_1 - Q_1 P_2)} - \\ - \frac{bP_1}{s^2(Q_2 P_1 - Q_1 P_2)},$$

where Q_j and P_j are determined by equations (1.13) and (1.14): Consequently the solutions (2.1) and (2.2) can be rewritten.

$$t_L(x, s) \multimap \frac{t_0}{s} = (t_c^0 - t_0) \times$$

$$\times \left\{\frac{P_1 + \varepsilon \mathrm{Ko}(Q_1 - K_1 P_1)}{s(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x - \right.$$

$$\left. - \frac{P_2 + \varepsilon \mathrm{Ko}(Q_2 - K_1 P_2)}{s(Q_2 P_1 - Q_1 P_2)} \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x \right\} +$$

$$+ b \frac{P_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x - P_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x}{s^2(Q_2 P_1 - Q_1 P_2)} = \Pi_1 + \Pi_2 , \quad (2.6)$$

$$\frac{\theta_0}{s} - \theta_L(x, s) = (\theta_0 - \theta_p) \times$$

$$\times \left\{\frac{Q_2 + \frac{1}{\varepsilon \mathrm{Ko}} P_2 - K_1 P_2}{s(Q_2 P_1 - Q_1 P_2)} (1 - \nu_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x - \right.$$

$$- \frac{Q_1 + \frac{1}{\varepsilon \mathrm{Ko}} P_1 - K_1 P_1}{s(Q_2 P_1 - Q_1 P_2)} (1 - \nu_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x \Bigg\} +$$

$$+ \frac{c}{\varepsilon \varrho} b \frac{P_1(1 - \nu_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x - P_2(1 - \nu_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x}{s^2(Q_2 P_1 - Q_1 P_2)}$$

$$= N_1 + N_2 . \tag{2.7}$$

The inversions of the first components $L^{-1}[II_1]$ and $L^{-1}[N_1]$ were found in § 1 a.

Here we will consider particularly the determination of the inversions of the remaining expressions π_2 and N_2, which represent the influence of a linear change of the temperature of the medium on the general solution. For the transfer potential of heat the initial expression has the form

$$\Pi_2 = b \frac{P_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x - P_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x}{s^2(Q_2 P_1 - Q_1 P_2)}$$

$$= \frac{b \Phi_{01}(s)}{s^2 \varphi_0(s)} . \tag{2.8}$$

We shall find the original of Π_2 by means of the theorem of expansion. We determine the roots of the denominator of (2.8), $s^2 \varphi_0(s)$. We have:

(a) a double zero root,

(b) the roots $s = s_n$ satisfying the equation $\varphi_0(s) = Q_2 P_1 - Q_1 P_2 = 0$. Then for (2.8) the expansion theorem can be written as follows

$$L^{-1}\left[\frac{b \Phi_{01}(s)}{s^2 \varphi_0(s)}\right]$$

$$= b \lim_{s \to 0} \left[\frac{\tau e^{s\tau} \Phi_{01}(s)}{\varphi_0(s)} + \frac{e^{s\tau} \Phi'_{01}(s)}{\varphi_0(s)} - \frac{e^{s\tau} \Phi_{01}(s) \varphi'_0(s)}{[\varphi_0(s)]^2}\right] +$$

$$+ \sum_{n=1}^{\infty} b \frac{\Phi_{01}(s_n)}{[s_n^2 \varphi_0(s_n)]'} e^{s_n \tau} . \tag{2.9}$$

Since

$$\varphi_0(0) = \nu_2^2 - \nu_1^2 ,$$

$$\varphi_0'(0) = \frac{R^2}{2\,a_q}(\nu_2^2 - \nu_1^2) \times$$

$$\times \left[1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}} + \frac{2}{\mathrm{Bi}_q} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}\right],$$

$$\Phi_{01}(0) = -(\nu_2^2 - \nu_1^2),$$

$$\Phi_{01}'(0) = -(\nu_2^2 - \nu_1^2)\frac{R^2}{2\,a_q} \times$$

$$\times \left(\frac{1}{\mathrm{Lu}} + \mathrm{Fe} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}} + \frac{x^2}{R^2}\right),$$

$$[s_n^2\,\varphi_0(s_n)]' = -\frac{a_q}{2\,R^2}\mu_n^3\,\psi_n,$$

$$\Phi_{01}(s_n) = P_{n2}\cos\nu_1\,\mu_n\frac{x}{R} - P_{n1}\cos\nu_2\,\mu_n\frac{x}{R},$$

the inversion of the transform (2.9) will have the form

$$L^{-1}[\Pi_2] = -b\,\tau + b\,\frac{R^2}{2\,a_q}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) -$$

$$- 2\sum_{n=1}^{\infty} b\frac{R^2}{a_q\,\mu_n^2}\,\frac{P_{n2}\cos\nu_1\,\mu_n\frac{x}{R} - P_{n1}\cos\nu_2\,\mu_n\frac{x}{R}}{\mu_n\,\psi_n} \times$$

$$\times \exp(-\mu_n^2\,\mathrm{Fo}) = (t_c^0 - t_0)\left\{-\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \right.\right.\right.$$

$$\left.\left.\left. + \frac{2}{\mathrm{Bi}_q}\right)\right] - 2\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\mu_n^2}\,\frac{P_{n2}\cos\nu_1\,\mu_n\frac{x}{R} - P_{n1}\cos\nu_2\,\mu_n\frac{x}{R}}{\mu_n\,\psi_n} \times\right.$$

$$\left. \times \exp(-\mu_n^2\,\mathrm{Fo})\right\}. \qquad (2.10)$$

In the original (2.10) a new criterion of similarity is introduced:

$$\mathrm{Pd} = \frac{bR^2}{a_q(t_c^0 - t_0)} \qquad (2.11)$$

representing the rate of change of temperature of the medium. More detailed analysis shows that this non-dimensional complex is a special case of the expression $(dT_a/dF_0)_{\max}$, which is called the Predvoditelev criterion. Hence (2.11)

is called the Predvoditelev criterion for the special case of the linear dependence of the temperature of the medium on the time. Of the quantities which enter into Pd, the constant b denotes the rate of change of the temperature of the medium, and the constant t_c^0 denotes the temperature of the medium at the initial moment of time.

We have now the inverse of the transform N_2:

$$N_2 = \frac{c}{\varepsilon\varrho} b \times$$

$$\times \frac{P_1(1-r_2^2)\cosh\sqrt{\left(\frac{s}{a_q}\right)} r_2 x - P_2(1-r_1^2)\cosh\sqrt{\left(\frac{s}{a_q}\right)} r_1 x}{s^2(Q_2 P_1 - Q_1 P_2)}$$

$$= \frac{c}{\varepsilon\varrho} b \frac{\Phi_{02}(s)}{s^2 \varphi_0(s)}. \qquad (2.12)$$

The roots of the denominator in (2.12) remain as before.
Hence

$$L^{-1}\left[\frac{c}{\varepsilon\varrho} b \frac{\Phi_{02}(s)}{s^2\varphi_0(s)}\right] = \frac{c}{\varepsilon\varrho} b \lim_{s\to 0}\left[\tau e^{s\tau}\frac{\Phi_{02}(s)}{\varphi_0(s)} + \right.$$

$$+ e^{s\tau}\frac{\Phi'_{02}(s)}{\varphi_0(s)} - e^{s\tau}\frac{\Phi_{02}(s)\varphi'_0(s)}{[\varphi_0(s)]^2} + \frac{c}{\varepsilon\varrho} b \times$$

$$\times \sum_{n=1}^{\infty}{}' \frac{\Phi_{02}(s_n)}{[s_n^2\varphi_0(s_n)]'} \exp(s_n\tau) = \frac{c}{\varepsilon\varrho} b \,\mathrm{Fe}\frac{R^2}{2a_q} \times$$

$$\times\left(1-\frac{x^2}{R^2}\right) - \sum_{n=1}^{\infty} 2\frac{c}{\varepsilon\varrho}\frac{bR^2}{a_q\mu_n^2} \times$$

$$\times \frac{P_{n1}(1-r_2^2)\cos r_2\mu_n\frac{x}{R} - P_{n2}(1-r_1^2)\cos r_1\mu_n\frac{x}{R}}{\mu_n\nu_n} \times$$

$$\times \exp(-\mu_n^2\,\mathrm{Fo}) = (\theta_0-\theta_p)\left\{\frac{1}{2}\mathrm{Pn}\,\mathrm{Pd}\left(1-\frac{x^2}{R_2}\right) - \frac{2}{\varepsilon\,\mathrm{Ko}} \times\right.$$

$$\times \sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\mu_n^2}\frac{P_{n1}(1-r_2^2)\cos r_2\mu_n\frac{x}{R} - P_{n2}(1-r_1^2)\cos r_1\mu_n\frac{x}{R}}{\mu_n\nu_n} \times$$

$$\left.\times \exp(-\mu_n^2\,\mathrm{Fo})\right\}, \qquad (2.13)$$

since

$$\Phi_{02}(0) = 0,$$

$$\Phi'_{02}(0) = -(\nu_2^2 - \nu_1^2)\,\mathrm{Fe}\,\frac{R^2}{2a_q}\left(\frac{x^2}{R^2} - 1\right),$$

$$\Phi_{02}(s_n) = P_{n1}(1-\nu_2^2)\cos\nu_2\,\mu_n\frac{x}{R} - P_{n2}(1-\nu_1^2)\times$$

$$\times\cos\nu_1\,\mu_n\frac{x}{R}.$$

That is, the final solution of our problem will have the form

$$T = \frac{t(x,\tau)-t_0}{t_c^0-t_0} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right] -$$

$$- \sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\,\mu_n\frac{x}{R} - C_{n1}\cos\nu_2\,\mu_n\frac{x}{R}\right)\times$$

$$\times\exp\left(-\mu_n^2\,\mathrm{Fo}\right); \qquad (2.14)$$

$$\Theta = \frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p} = 1 - \frac{1}{2}\,\mathrm{Pn}\,\mathrm{Pd}\left(1-\frac{x^2}{R^2}\right) -$$

$$- \sum_{n=1}^{\infty}\left[C_{n1}^*(1-\nu_2^2)\cos\nu_2\,\mu_n\frac{x}{R} - C_{n2}^*\times\right.$$

$$\left.\times(1-\nu_1^2)\cos\nu_1\,\mu_n\frac{x}{R}\right]\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (2.15)$$

where

$$C_{nj} = 2\,\frac{\left(1-\varepsilon\,\mathrm{Ko}\,K_1 + \dfrac{\mathrm{Pd}}{\mu_n^2}\right)P_{nj} + \varepsilon\,\mathrm{Ko}\,Q_{nj}}{\mu_n\varphi_n},$$

$$C_{nj}^* = \frac{1}{\varepsilon\,\mathrm{Ko}}\,C_{nj}. \qquad (2.16)$$

Q_{nj}, P_{nj} and φ_n are determined by equations (1.21)–(1.23) and the roots μ_n by the characteristic equation (1.20). The criteria of similarity remain the same as in the problems $t_a = \text{const}$. with the substitution of t_a^0 for t_a.

The mean non-dimensional temperature and potential of transfer of mass are obtained in the ordinary way. They will be:

$$\overline{T} = \frac{\bar{t}(\tau) - t_0}{t_c^0 - t_0} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \left(\frac{1}{3} + \frac{1}{\mathrm{Bi}_q}\right)\right] - \\ - \sum_{n=1}^{\infty} D_n \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (2.17)$$

$$\overline{\Theta} = \frac{\theta_0 - \theta(\tau)}{\theta_0 - \theta_p} = 1 + \frac{1}{3}\mathrm{Pn}\,\mathrm{Pd} - \\ - \sum_{n=1}^{\infty} D_n^* \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (2.18)$$

where

$$D_n = D_{n2} - D_{n1},\ D_n^* = D_{n1}^* - D_{n2}^*.$$

The expressions for D_{nj} and D_{nj}^* remain the same as in the other mean problems for an infinite plate, taking account of equation (2.16).

The local and mean non-dimensional rates of change of transfer potentials will appear like:

$$\frac{dT}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty}\left(C_{n2}\cos \nu_1 \mu_n \frac{x}{R} - C_{n1}\cos \nu_2 \mu_n \frac{x}{R}\right) \times \\ \times \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right) - \mathrm{Pd}, \qquad (2.19)$$

$$\frac{d\overline{T}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right) - \mathrm{Pd}. \qquad (2.20)$$

$$\frac{d\Theta}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty}\left[C_{n1}^*(1-\nu_2^2)\cos \nu_2 \mu_n \frac{x}{R} - C_{n2}^* \times \right. \\ \left. \times (1-\nu_1^2)\cos \nu_1 \mu_n \frac{x}{R}\right] \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (2.21)$$

$$\frac{d\overline{\Theta}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^* \mu_n^2 \exp\left(-\mu_n^2 \mathrm{Fo}\right). \qquad (2.22)$$

Analysis of the solutions (2.14), (2.15) and (2.17) — (2.22) shows that the initial value of Fo for simplified calculations (the possibility of using two terms of the infinite series) up to Pd $\approx 0{\cdot}25$ remains approximately the same as in a medium with constant temperature (see Table 18). With subsequent increase in Pd, the initial values of Fo slowly diminish.

TABLE 18. INITIAL VALUES OF Fo FOR SIMPLIFIED CALCULATION OF HEAT AND MASS TRANSFER IN AN INFINITE PLATE FOR $t_a = t_n^0 - b\tau$

Index		Fo, $\lvert\Delta\rvert$% 1·0%	Fo, $\lvert\Delta\rvert$% 0·5%
$1-T$	$x/R=0$	0·73	0·88
	$x/R=1$	0·93	1·09
$1-\Theta$	$x/R=0$	0·79	0·95
	$x/R=1$	1·04	
$1-\overline{\Theta}$		0·12	0·17
$1-\overline{\Theta}$		0·14	0·30
$d\overline{T}/d\mathrm{Fo}$		0·26	0·35
$d\overline{\Theta}/d\mathrm{Fo}$		1·04	

With increase in the non-dimensional time the influence of the infinite series on the values of the non-dimensional potentials also diminishes. Thus, for T, Pd $= 0{\cdot}25$, $\mathrm{Bi}_q = 10$. Thus, in the expression for T when Pd $= 0{\cdot}25$, $\mathrm{Bi}_q = 10$, $x/R = 0$, $\varepsilon = 0{\cdot}5$, Pn $= 0{\cdot}5$, Ko $= 1{\cdot}2$, $\mathrm{Bi}_m = 10$, the term of the infinite sum has, in comparison with the absolute term, the following percentage values:

for Fo = 5·0,	23·7	per cent
for Fo = 7·0,	1·31	per cent
for Fo = 10·0	0·110	per cent

Beginning from a value of $\mathrm{Fo} = \mathrm{Fo}_{\mathrm{quasist}}$ the non-dimensional potentials are determined exclusively by the steady state terms. Then the temperature at any point of the plate becomes

a linear function of the time, and the temperature distribution through the thickness of the material becomes parabolic. Hence, starting from $\mathrm{Fo}_{quasist}$, we shall have the same distribution of temperature in space. Such a state of affairs is called quasi-stationary. At the same time the redistribution of mass potential within the body ceases. A stationary parabolic distribution of mass potential is established. As the criterion Pd increases, the quasi-stationary condition of heating (or cooling) of the body is set up at lower values of Fo.

Thus, for $\mathrm{Fo} \geqslant \mathrm{Fo}_{\mathrm{quasist}}$ equations (2.14), (2.15), (2.17) and (2.18) take the form:

$$T = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right], \qquad (2.14')$$

$$\Theta = 1 + \frac{1}{2}\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{x^2}{R^2}\right), \qquad (2.15')$$

$$\bar{T} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \left(\frac{1}{3} + \frac{1}{\mathrm{Bi}_q}\right)\right], \qquad (2.17')$$

$$\bar{\Theta} = 1 + \frac{1}{3}\mathrm{Pn}\,\mathrm{Pd}. \qquad (2.18')$$

The connection between the coefficients of heat and mass transfer C_{nj}, C^*_{nj}, D_n and D^*_n and the Predvoditelev criterion is shown in Tables 19–22.

(b) *Sphere. Constant initial conditions* ($\Gamma = 2$, $\xi = r$, $t(r, 0) = t_0 = \mathrm{const.}$, $\theta(r, 0) = \theta_0 = \mathrm{const.}$, $t_a = t_a^0 - b\tau$). The general solution of the problem in transform can be written:

$$\vartheta_L(r, s) - \frac{rt_0}{s} = B_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r + B_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r,$$

$$u_L(r, s) - \frac{r\theta_0}{s} = B_1 \frac{c}{cQ}(1 - \nu_1^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 r +$$

$$+ B_2 \frac{c}{cQ}(1 - \nu_2^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 r. \qquad (2.23)$$

TABLE 19. RELATION OF THE COEFFICIENTS OF HEAT AND

Pd	C_{n1}		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0·3080	—0·01452	0·04880	0·9378	—0·5134	—0·02420
0·1	—0·3152	—0·01532	0·04976	0·9908	—0·5254	—0·02554
0·2	—0·3225	—0·01612	0·05073	1·044	—0·5376	—0·02687
0·25	—0·3261	—0·01652	0·05125	1·070	—0·5436	—0·02754
0·3	—0·3297	—0·01692	0·05169	1·097	—0·5496	—0·02821
0·4	—0·3369	—0·01772	0·05266	1·150	—0·5616	—0·02954
0·5	—0·3441	—0·01852	0·05362	1·203	—0·5736	—0·03087
0·6	—0·3514	—0·01932	0·05458	1·256	—0·5858	—0·03221
0·7	—0·3586	—0·02012	0·05555	1·309	—0·5978	—0·03354
0·75	—0·3622	—0·02052	0·05603	1·355	—0·6038	—0·03421
0·8	—0·3658	—0·02092	0·05651	1·362	—0·6098	—0·03487
0·9	—0·3730	—0·02172	0·05748	1·414	—0·6218	—0·03621
1·0	—0·3803	—0·02252	0·05844	1·467	—0·6340	—0·03754

† Calculation carried out for $\mathrm{Lu} = 0{\cdot}3$; $\varepsilon = 0{\cdot}5$; $\mathrm{Ko} = 1{\cdot}2$;

TABLE 20. RELATION OF THE COEFFICIENTS OF HEAT AND

Pd	C_{m}		C_{n2}		C^{**}_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—1·135	—0·003402	0·1399	0·03438	—1·892	—0·00572
0·25	—1·222	—0·007787	0·1498	0·08778	—2·037	—0·01298
0·50	—1·309	—0·01217	0·1596	0·1412	—2·182	—0·02029
0·75	—1·396	—0·01656	0·1695	0·1946	—2·327	—0·02761
1·00	—1·483	—0·02094	0·1793	0·2480	—2·472	—0·03491

$\mathrm{Lu} = 1{\cdot}0$, $\varepsilon = 0{\cdot}5$, $\mathrm{Ko} = 1{\cdot}2$,

TABLE 21. RELATION OF THE COEFFICIENTS OF HEAT AND MASS
$\mathrm{Pn} = 0{\cdot}1667$,

Pd	C_{n1}		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0·3188	—0·005276	0·04618	0·9158	—0·5312	—0·008794
0·25	—0·3258	—0·006395	0·04536	1·062	—0·5431	—0·01066
0·50	—0·3327	—0·006914	0·04455	1·207	—0·5546	—0·01153
0·75	—0·3397	—0·007733	0·04373	1·353	—0·5663	—0·01289
1·00	—0·3466	—0·008552	0·04292	1·499	—0·5778	—0·01426

MASS TRANSFER TO Pd $= bR^2/a_q(t_c^0 - t_0)$ (INFINITE PLATE†)

C_{n2}^*		D_n		D_n^*	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·08130	1·564	0·2570	0·6500	0·9616	—0·1137
0·08295	1·652	0·2628	0·6867	0·9842	—0·1202
0·08457	1·749	0·2687	0·7235	1·007	—0·1265
0·08513	1·784	0·2716	0·7416	1·018	—0·1297
0·08617	1·829	0·2745	0·7603	1·029	—0·1330
0·08778	1·917	0·2804	0·7971	1·052	—0·1394
0·08938	2·005	0·2862	0·8338	1·075	—0·1458
0·09098	2·094	0·2921	0·8706	1·098	—0·1523
0·09260	2·182	0·2980	0·9072	1·120	—0·1587
0·09340	2·225	0·3009	0·9252	1·132	—0·1618
0·09420	2·270	0·3039	0·9440	1·143	—0·1650
0·09582	2·357	0·3097	0·9800	1·165	—0·1714
0·09742	2·445	0·3156	1·031	1·88	—0·1778

$Pn = 0·5$; $Bi_m = 10$; $Bi_q = 10$ Constant Initial Conditions.

MASS TRANSFER TO Pd $= bR^2a_q(t_c^0 - t_0)$ (INFINITE PLATE)

C_{n2}^*		D_n		D_n^*	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·2334	0·0572	0·9009	0·02439	0·8436	—0·01538
0·2497	0·1463	0·9692	0·06206	0·909	—0·03962
0·2661	0·2354	1·037	0·09975	0·974	—0·06386
0·2826	0·3244	1·106	0·1374	1·039	—0·08806
0·2989	0·4134	1·174	0·1751	1·104	—0·1123

$Pn = 0·5$, $Bi_m = Bi_q = 10$.

TRANSFER TO Pd (INFINITE PLATE) $Lu = 0·3$, $\varepsilon = 0·5$, $Ko = 1·2$, $Bi_m = Bi_q = 10$

C_{n2}		D_n		D_n^*	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·07694	1·527	0·2620	0·6353	0·9045	—0·03952
0·07562	1·770	0·2662	0·7368	0·9248	—0·04563
0·07426	2·012	0·2702	0·8372	0·9446	—0·05209
0·07290	2·255	0·2743	0·9380	0·9645	—0·05839
0·07155	2·499	0·2783	1·040	0·9843	—0·06471

TABLE 22. RELATION OF THE COEFFICIENTS OF HEAT (INFINITE PLATE) WHEN THE TEMPERATURE OF THE

$\mathrm{Ko} = 1{\cdot}2$, $\mathrm{Pn} = 0{\cdot}5$,

Bi_q	O_{n1}		O_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5	—0·3295	—0·00990	0·1643	1·010	—0·5493	—0·01650
7	—0·3274	—0·01266	0·09726	1·054	—0·5458	—0·02110
10	—0·3261	—0·01652	0·05125	1·070	—0·5436	—0·02754
15	—0·3251	—0·01773	0·01597	1·079	—0·5419	—0·02956
20	—0·3243	—0·01860	—0·003319	1·088	—0·5406	—0·03101

The constants B_j are determined from the boundary conditions (III–IV), which in transformation have the form (2.4) and (2.5). Adjusting (2.22) and (2.23) to the boundary conditions, we solve the system of equations thus obtained for B_1 and B_2. Substituting the values thus found in (2.22) and (2.23) we have:

$$\vartheta_L(\tau, s) - \frac{\tau t_0}{s} = (t_c^0 - t_0) \times$$

$$\times \left\{ \frac{\tilde{P}_1 + \varepsilon \mathrm{Ko}(\tilde{Q}_1 - K_1 \tilde{P}_1)}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 \tau - \right.$$

$$\left. - \frac{\tilde{P}_2 + \varepsilon \mathrm{Ko}(\tilde{Q}_2 - K_1 \tilde{P}_2)}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 \tau \right\} +$$

$$+ b \frac{\tilde{P}_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 \tau - \tilde{P}_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 \tau}{s^2(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} = \Pi_1 + \Pi_2, \quad (2.24)$$

$$\frac{\tau \theta_0}{s} - u_L(\tau, s) = (\theta_0 - \theta_p) \times$$

$$\times \left\{ \frac{\tilde{Q}_2 + \frac{1}{\varepsilon \mathrm{Ko}} \tilde{P}_2 - K_1 \tilde{P}_2}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} (1 - \nu_1^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 \tau - \right.$$

$$\left. - \frac{\tilde{Q}_1 + \frac{1}{\varepsilon \mathrm{Ko}} \tilde{P}_1 - K_1 \tilde{P}_1}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} (1 - \nu_2^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 \tau \right\} -$$

AND MASS TRANSFER TO THE BIOT CRITERION Bi_q MEDIUM IS FALLING LINEARLY $\mathrm{Lu} = 0{\cdot}3$, $\varepsilon = 0{\cdot}5$, $\mathrm{Bi}_m = 10$, $\mathrm{Pd} = 0{\cdot}25$

$C^*_{n_2}$		D_n		D^*_n	
μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0·2739	1·684	0·3789	0·7390	1·013	—0·1278
0·1621	1·757	0·3152	0·7500	1·016	—0·1303
0·08543	1·784	0·2716	0·7416	1·018	—0·1297
0·02662	1·799	0·2383	0·7332	1·019	—0·1299
0·005533	1·814	0·2197	0·7314	1·019	—0·1305

$$-b\frac{c}{\varepsilon\varrho}\,\frac{\tilde{P}_2(1-\nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 r-\tilde{P}_1(1-\nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 r}{S^2(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)}$$
$$=N_1+N_2\,. \tag{2.25}$$

$\tilde{Q}_j$ and $\tilde{P}_j$ are determined from equations (1.60) and (1.61), and ν_j from equation (2.3). The inversions of Π_1 and N_1 are found in Section 1b.

Let us determine the inversions of the expressions Π_2 and N_2:

$$\Pi_2 = b\,\frac{\tilde{P}_2\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 r-\tilde{P}_1\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 r}{s^2(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)}=b\,\frac{\Phi_1(s)}{\varphi_1(s)}\,. \tag{2.26}$$

As usual, for the sphere the numerator and denominator in (2.26) are not generalized, but may be made so by dividing both parts of (2.26) by s/a_q, i.e. $\Phi_1(s)/(s/a_q)=\Phi_{01}(s)$, $\varphi_1(s)/(s/a_q)=\varphi_0(s)$. In view of the fact that the roots of the denominator are the double zero root and the roots $s_n=-a_q\,\mu_n^2/R^2$, corresponding to the equation $\varphi_0(s)/s^2=\varphi_0(s)=0$, the generalized expansion theorem has the form:

$$L^{-1}[\Pi_2]=L^{-1}\left[b\frac{\Phi_1(s)}{\varphi_1(s)}\right]$$
$$=b\lim_{s\to 0}\left|\tau e^{s\tau}\frac{\Phi_{01}(s)}{\varphi_0(s)}+e^{s\tau}\frac{\Phi'_{01}(s)}{\varphi_0(s)}-e^{s\tau}\frac{\Phi_{01}(s)\,\varphi'_0(s)}{[\varphi_0(s)]^2}\right]+$$

$$+\sum_{n=1}^{\infty} b\frac{\Phi_1(s_n)}{\varphi_1'(s_n)}\exp(s_n\tau)$$

$$=-b\tau r+b\frac{R^2}{6a_q}r\left(1-\frac{r^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)+$$

$$+\sum_{n=1}^{\infty}2\frac{b}{a_q}\frac{R^2}{\mu_n^2}\left(\frac{P_{n2}R\sin\nu_1\mu_n\frac{r}{R}}{\mu_n\varphi_n}-\right.$$

$$\left.-\frac{P_{n1}R\sin\nu_2\mu_n\frac{r}{R}}{\mu_n\varphi_n}\right)\exp(-\mu_n^2\,\mathrm{Fo})$$

$$=\tau(t_c^0-t_0)\left\{-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{6}\left(1-\frac{r^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right]-\right.$$

$$-2\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\mu_n^2}\frac{P_{n1}R\sin\nu_2\mu_n\frac{r}{R}-P_{n2}R\sin\nu_1\mu_n\frac{r}{R}}{\mu_n\varphi_n}\times$$

$$\left.\times\exp(-\mu_n^2\,\mathrm{Fo})\right\},\qquad(2.27)$$

since $\varphi_0(0)=(\nu_2^2-\nu_1^2)\nu_1\nu_2$,

$$\varphi_0'(0)=\frac{R^2}{6a_q}\nu_1\nu_2\left(1+\frac{1}{\mathrm{Lu}}+\mathrm{Fe}+\right.$$

$$\left.+\frac{2}{\mathrm{Bi}_q}+\frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}\right)(\nu_2^2-\nu_1^2),$$

$$\Phi_{01}(0)=-\tau(\nu_2^2-\nu_1^2)\nu_1\nu_2,$$

$$\Phi_{01}'(0)=\tau\frac{R^2}{6a_q}\nu_1\nu_2\times$$

$$\times\left(-\frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}-\frac{1}{\mathrm{Lu}}-\mathrm{Fe}-\frac{r^2}{R^2}\right)(\nu_2^2-\nu_1^2),$$

$$\varphi_1'(s_n)=-\frac{a_q\mu_n^3}{2R^4}\varphi_n,$$

$$\Phi_1(s_n)=-\frac{1}{R}\left(P_{n2}\sin\nu_1\mu_n\frac{r}{R}-P_{n1}\sin\nu_2\mu_n\frac{r}{R}\right),$$

the original

$$N_2 = -\,b\,\frac{c}{\varepsilon\varrho}\times$$

$$\times\,\frac{\tilde{P}_2(1-\nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,r_1\,r-\tilde{P}_1(1-\nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\,r_2\,r}{s^2(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)}$$

$$=-\frac{bc}{\varepsilon\varrho}\,\frac{\varphi_2(s)}{\varphi_1(s)}$$

can be obtained in the same way as for the inversion of Π_2. Calculation gives

$$L^{-1}[N_2]=\frac{c}{\varepsilon\varrho}\,b\,\frac{R^2}{6\,a_q}\,r\,\mathrm{Fe}\left(1-\frac{r^2}{R^2}\right)-$$

$$-2\sum_{n=1}^{\infty}\frac{c}{\varepsilon\varrho}\,b\,\frac{R^2}{a_q\,\mu_n^2}\times$$

$$\times\,\frac{P_{n2}(1-\nu_1^2)\,R\sin\nu_1\,\mu_n\,\frac{r}{R}-P_{n1}(1-\nu_2^2)\,R\sin\nu_2\,\mu_n\,\frac{r}{R}}{\mu_n\varphi_n}\,e^{-\mu_n^2\,\mathrm{Fo}}$$

$$=(\theta_0-\theta_p)\,r\left\{\frac{1}{2}\,\mathrm{Pn}\,\mathrm{Pd}\left(1-\frac{r^2}{R^2}\right)-2\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\varepsilon\,\mathrm{Ko}\,\mu_n^2}\times\right.$$

$$\times\left[\frac{P_{n2}(1-\nu_1^2)\sin\nu_1\,\mu_n\frac{r}{R}-P_{n1}(1-\nu_2^2)\sin\nu_2\,\mu_n\frac{r}{R}}{\frac{r}{R}\mu_n\varphi_n}\right]\times$$

$$\times\exp\left(-\mu_1^2\,\mathrm{Fo}\right).\qquad(2.28)$$

Thus the solution of the system of differential equations for the heat and mass transfer potentials of heat and mass, using the inversion (2.27) and (2.28) and also the solutions of § 1 (b), can be written:

$$T=\frac{t(r,\tau)-t_0}{t_c^0-t_0}=1-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{6}\left(1-\frac{r^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}\left(C_{n1}\,\frac{\sin\nu_2\,\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{n2}\,\frac{\sin\nu_1\,\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\mu_n^2\,\mathrm{Fo}\right),\quad(2.29)$$

$$\Theta = \frac{\theta_0 - \theta(r, \tau)}{\theta_0 - \theta_p} = 1 + \frac{1}{6}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right) -$$

$$- \sum_{n=1}^{\infty}\left[C^*_{n2}(1 - \nu_1^2)\,\frac{\sin \nu_1 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} -\right.$$

$$\left. - C^*_{n1}(1 - \nu_2^2)\,\frac{\sin \nu_2 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}}\right] \exp\left(-\mu_n^2\,\mathrm{Fo}\right). \qquad (2.30)$$

Here C_{nj} and C^*_{nj} are determined by equation (2.16); Q_{nj}, P_{nj} and ψ_n by equations (1.66), (1.67) and (1.70).

The mean non-dimensional potentials for a sphere will be expressed by equations (2.31) and (2.32):

$$\overline{T} = \frac{\bar{t}(\tau) - t_0}{t_c^0 - t_0} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{15}\left(1 + \frac{5}{\mathrm{Bi}_q}\right)\right] -$$

$$- \sum_{n=1}^{\infty} D_n \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (2.31)$$

$$\bar{\bar{\Theta}} = \frac{\theta_0 - \bar{\bar{\theta}}(\tau)}{\theta_0 - \theta_p} = 1 + \frac{1}{15}\,\mathrm{Pn}\,\mathrm{Pd} -$$

$$- \sum_{n=1}^{\infty} D_n^* \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (2.32)$$

where $D_n = D_{n1} - D_{n2}$, $D_n^* = D_{n2}^* - D_{n1}^*$, D_{nj} and D_{nj}^* have the usual form for a sphere, taking account of the new expression for C_{nj} and C^*_{nj}.

The characteristic roots μ_n in equations (2.27) – (2.32) remain the same as in problems where $t_a = \mathrm{const}$.

From the solutions for the sphere, and for a plate, it follows that the non-dimensional temperature and transfer potential of mass at any point of the body are proportional to the Predvoditelev criterion and consequently to the rate of change of the temperature of the medium. In view of the comparatively rapid convergence of the infinite series, beginning from $\mathrm{Fo} = \mathrm{Fo}_{quasist}$ this criterion may be neglected. In this case the distribution of potentials in the material becomes para-

bolic, and the change of T with time is linear (for $\mathrm{Bi}_q = \text{const.}$), i.e.

$$\text{for } \mathrm{Fo} \geqslant \mathrm{Fo}_{quasist}$$

$$T = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right], \qquad (2.29')$$

$$\Theta = 1 + \frac{1}{6}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right), \qquad (2.30')$$

$$\bar{T} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{15}\left(1 + \frac{5}{\mathrm{Bi}_q}\right)\right], \qquad (2.31')$$

$$\bar{\Theta} = 1 + \frac{1}{15}\,\mathrm{Pn}\,\mathrm{Pd}\,. \qquad (2.32')$$

(c) *Infinite plate. Parabolic initial conditions*

$$\left[\Gamma = 0\,,\ \xi = x\,,\ t(x,0) = t_c + \left(\frac{x}{R}\right)^2 (t_u - t_c)\,,\right.$$

$$\left.\theta(x,0) = \theta_c - \left(\frac{x}{R}\right)^2 (\theta_c - \theta_u)\,,\ t_c = t_c^0 - b\tau\right].$$

The general solution of the system of differential equations of heat and mass transfer in transformation with parabolic initial conditions is as follows:

$$t_L(x,s) = \frac{t_u - (t_u - t_n)\frac{x^2}{R^2}}{s} + \frac{2\,a_m\,\omega}{\delta^2} +$$

$$+ B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)}\, r_1\, x + B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)}\, r_2\, x\,, \qquad (2.33)$$

$$\theta_L(x,s) = \frac{\theta_u - (\theta_u - \theta_n)\frac{x^2}{R^2}}{s} + \frac{2\,\frac{a_q\,c}{\varepsilon\varrho}\left(\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right)}{\delta^2} +$$

$$+ B_1\,\frac{c}{\varepsilon\varrho}\,(1 - r_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)}\, r_1\, x +$$

$$+ B_2\,\frac{c}{\varepsilon\varrho}\,(1 - r_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)}\, r_2\, x\,. \qquad (2.34)$$

In equations (2.33) and (2.34), B_1 and B_2 are constants with respect to the co-ordinates. r_j is expressed in the usual way and ω is determined by equation (1.91).

B_j, found from the boundary conditions (2.4) and (2.5), have the form:

$$B_1 = -\left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2' P_2 - \Omega_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right],$$

$$B_2 = \frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2' P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)},$$

where Q_j, P_j and Ω_k have the values set out in § 1 (c),

$$\Omega_2' = -2a_q\left[K_1\left(\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right) + \omega \mathrm{Lu} + \frac{b}{2a_q}\right].$$

Hence the solution of the system of equations (I–VI) in transform will be:

$$t_L(x, s) = \frac{t_u - (t_u - t_n)\frac{x^2}{R^2}}{s} + \frac{2a_m\omega}{s^2} + \left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \right.\right.$$

$$\left. + \frac{\Omega_2' P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x - \left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \right.$$

$$\left.\left. + \frac{\Omega_2' P_2 - \Omega_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x\right\}, \qquad (2.35)$$

$$\theta_L(x, s) = \frac{\theta_u - (\theta_u - \theta_n)\frac{x^2}{R^2}}{s} + \frac{2\frac{a_q c}{\varepsilon\varrho}\left(\frac{t_u - t_n}{R^2} + \omega \mathrm{Lu}\right)}{s^2} +$$

$$+ \left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2' P_1 - \Omega_4 Q_1}{s^2(Q_2 P_1 - Q_1 P_2)}\right]\frac{c}{\varepsilon\varrho}(1 - v_2^2) \times\right.$$

$$\times \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 x - \left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2' P_2 - \Omega_4 Q_2}{s^2(Q_2 P_1 - Q_1 P_2)}\right] \times$$

$$\left. \times \frac{c}{\varepsilon\varrho}(1 - v_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 x\right\}. \qquad (2.36)$$

The originals of the principal expressions in equations (2.35) and (2.36) were obtained in § 1(c), and hence, substituting their values in the inverse transformations for $L^{-1}[t_L(x, s)]$ and $L^{-1}[\theta_L(x, s)]$ and making the necessary simplifications, we finally obtain:

$$T_1 = \frac{t(x, \tau) - t_n}{t_c^0 - t_n} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right] -$$

$$- \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - C_{n1} \cos \nu_2 \mu_n \frac{x}{R} \right) \exp \left(- \mu_n^2 \, \mathrm{Fo} \right), \quad (2.37)$$

$$\Theta_1 = \frac{\theta_n - \theta(x, \tau)}{\theta_n - \theta_p} = 1 + \frac{1}{2} \, \mathrm{Pd} \, \mathrm{Pn} \left(1 - \frac{x^2}{R^2} \right) -$$

$$- \sum_{n=1}^{\infty} \left[C_{n1}^* (1 - \nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} - C_{n2}^* (1 - \nu_1^2) \cos \nu_1 \mu_n \frac{x}{R} \right] \times$$

$$\times \exp \left(- \mu_n^2 \, \mathrm{Fo} \right), \quad (2.38)$$

where

$$C_{nj} = 2 \, \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \; C_{nj}^* = \frac{1}{\varepsilon \, \mathrm{Ko}} \, C_{nj}, \quad (2.39)$$

$$G_1 = 1 - 2 \, W \left(\frac{1}{\mathrm{Bi}_q} - \frac{1}{\mu_n^2} \right) - \varepsilon \, \mathrm{Ko} \, K_1 -$$

$$- 2 \, \varepsilon \, \mathrm{Ko} \, K_2 \, \frac{(K_1 + 1) \, \mathrm{Lu}}{\mu_n^2} + \frac{\mathrm{Pd}}{\mu_n^2}, \quad (2.40)$$

$$G_2 = 2 \, \varepsilon \, \mathrm{Ko} \, K_2 \left(\frac{1}{\mathrm{Bi}_m} - \frac{\mathrm{Lu}}{\mu_n^2} \right) - \varepsilon \, \mathrm{Ko} \, . \quad (2.41)$$

Expressions for the remaining quantities will be found in § 1 (c).

(d) *Sphere. Parabolic initial conditions* [$\Gamma' = 2$, $\xi = r$, $t(r, 0) = t_u + (r/R)^2 (t_n - t_u)$, $\theta(r, 0) = \theta_u - (r/R)^2 (\theta_u - \theta_n)$, $t_c = t_c - b\tau$)].

The solutions for the sphere in representation may be written in the following way:

$$\vartheta_L(r, s) = \frac{r \left[t_u - (t_u - t_n) \frac{r^2}{R^2} \right]}{s} + \frac{6 \, r a_m \omega}{s^2} +$$

$$+ \left\{ \left[\frac{\Omega_1 \widetilde{P}_1 - \Omega_2 \widetilde{Q}_1}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \frac{\Omega_2' \widetilde{P}_1 - \Omega_4 \widetilde{Q}_1}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] \times \right.$$

$$\times \sinh \sqrt{\left(\frac{s}{a_q} \right)} \nu_2 r - \left[\frac{\Omega_1 \widetilde{P}_2 - \Omega_3 \widetilde{Q}_2}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \right.$$

$$\left. \left. + \frac{\Omega_2' \widetilde{P}_2 - \Omega_4 \widetilde{Q}_2}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] \sinh \sqrt{\left(\frac{s}{a_q} \right)} \nu_1 r \right\}, \quad (2.42)$$

$$u_L(r,s) = \frac{r\left[\theta_u - (\theta_u - \theta_n)\frac{r^2}{R^2}\right]}{s} + \frac{c}{\varepsilon\varrho}\left\{\frac{6\,r a_q\left[\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right]}{s^2} + \right.$$

$$+\left[\frac{\Omega_1\tilde{P}_1 - \Omega_3\tilde{Q}_1}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} + \frac{\Omega_2'\tilde{P}_1 - \Omega_4\tilde{Q}_1}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right](1 - \nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 r -$$

$$-\left[\frac{\Omega_1\tilde{P}_2 - \Omega_3\tilde{Q}_2}{s(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)} + \frac{\Omega_2'\tilde{P}_2 - \Omega_4\tilde{Q}_2}{s^2(\tilde{Q}_2\tilde{P}_1 - \tilde{Q}_1\tilde{P}_2)}\right]\times$$

$$\left.\times\,(1 - \nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 r\right\}. \qquad (2.43)$$

Here

$$\Omega_2' = -\,6\,a_q\left[K_1\left(\frac{t_u - t_n}{R^2} + \omega\,\mathrm{Lu}\right) + \omega\,\mathrm{Lu} + \frac{b}{6\,a_\varsigma}\right],$$

the values of the remaining quantities are given in § 1 (d). $L^{-1}[\vartheta_L(r,s)]$ and $L^{-1}[u_L(r,s)]$ may be found by using the generalized expansion theorem. Since the inversions of the principal expressions in (2.42) and (2.43) have been obtained in § 1 (d), the final solution of the problem, after several transformations, will take the form:

$$T_1 = \frac{t(r,\tau) - t_n}{t_c^0 - t_n} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right] -$$

$$-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}} - C_{n2}\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (2.44)$$

$$\Theta_1 = \frac{\theta_n - \theta(r,\tau)}{\theta_n - \theta_p} = 1 + \frac{1}{6}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right) -$$

$$-\sum_{n=1}^{\infty}\left[C_{n2}^*(1 - \nu_1^2)\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}} - \right.$$

$$\left. - C_{n1}^*(1 - \nu_2^2)\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}\right]\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (2.45)$$

where

$$C_{nj} = 2\,\frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \Psi_n}\,, \quad C^*_{nj} = \frac{C_{nj}}{\varepsilon\,\mathrm{Ko}}\,, \tag{2.46}$$

$$G_1 = 1 - 2\,W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,K_1 -$$

$$- 6\,\varepsilon\,\mathrm{Ko}\,K_2\,\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2} + \frac{\mathrm{Pd}}{\mu_n^2}\,, \tag{2.47}$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,, \tag{2.48}$$

and μ_n is determined from the solutions of the characteristic equation

$$Q_{n2} P_{n1} - Q_{n1} P_{n2} = 0\,.$$

(c) *Some remarks on the method of determining the coefficients of heat and mass transfer.* The existence of the solutions (2.14′), (2.15′), (2.29′) and (2.30′) enable us to propose a method of determining the coefficients of transfer of heat and of associated matter (A. V. Luikov, 1935; E. P. Shurygina, 1941; Y. A. Mikhailov, 1956). The essence of this method is as follows. Beginning from $\mathrm{Fo} < \mathrm{Fo}_{\mathrm{quasist}}$, the determination of the linear change of temperature of the medium and the temperatures of two points in the moist body is carried out. At the instant in question the distribution of moisture in the body is found. On the basis of these data we calculate:

1. *The Predvoditelev criterion*

$$\mathrm{Pd} = \frac{AR^2}{\xi^2 - R^2}\,\frac{t(R,\tau) - t(\xi,\tau)}{t_c - t_0}\,,$$

where A is equal to 2 for a plate, 4 for a cylinder and 6 for a sphere, and ξ is an arbitrary coordinate of the moist body.

2. *The coefficient of temperature conductivity*

$$a_c = \frac{b}{A}\,\frac{\xi^2 - R^2}{t(R,\tau) - t(\xi,\tau)}\,.$$

3. *The Posnov criterion*

$$\mathrm{Pn} = \frac{A(\Theta - 1)}{\mathrm{Pd}}.$$

4. *The thermo-gradient coefficient (or Soret coefficient)*

$$\delta = \mathrm{Pn}\frac{\theta_0 - \theta_p}{t_c - t_0} = \mathrm{Pn}'\frac{u_0 - u_p}{t_c - t_0},$$

where θ_0 and θ_p are the initial and equilibrium values of the transfer potential of mass, u_0 and u_p the initial and equilibrium moisture content.

5. *The volume heat capacity*

$$c_q \gamma_0 = \frac{a A}{2 bR}[t_c - t(R, \tau)].$$

6. *The coefficient of thermal conductivity*

$$\lambda = c_q \gamma_0 a_q.$$

The advantage of this method is that all the above criteria of similarity and coefficients are determined in one experiment. Its disadvantage lies in the difficulty of maintaining an exact linear change in the temperature of the medium.

§ 3. Non-stationary Fields of Transfer Potential of Heat and Mass. Temperature of Medium — an Exponential Function of the Time

Let us consider the solution of the equations of heat and mass transfer (I)–(VI) for the case of the exponential variation of the temperature of the medium with time. Since the method of solving the system of equations (I – VI) when $t_a = t_a^0 e^{-x\tau}$ is essentially identical with that used for the problems discussed above, we append here the final results.

(a) *Solutions for an infinite plate. Constant initial conditions.*

$$T = \frac{t(x, \tau) - t_0}{t_c^0 - t_0} = - W_0 + W_c \times$$

$$\times \frac{\overline{P}_1 \cos\left(\nu_2 \sqrt{(\mathrm{Pd}')}\,\frac{x}{R}\right) - \overline{P}_2 \cos\left(\nu_1 \sqrt{(\mathrm{Pd}')}\,\frac{x}{R}\right)}{\overline{Q}_2\,\overline{P}_1 - \overline{Q}_1\,\overline{P}_2} \exp\left(-\mathrm{Pd}'\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - C_{n1} \cos \nu_2 \mu_n \frac{x}{R}\right) \times$$

$$\times \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (3.1)$$

$$\Theta = \frac{\theta_0 - \theta(x,\tau)}{\theta_0 - \theta_p} = 1 + \frac{W_c}{\varepsilon\,\mathrm{Ko}} \times$$

$$\times \frac{\overline{P}_2(1-\nu_1^2)\cos\left(\nu_1 \sqrt{(\mathrm{Pd}')}\,\frac{x}{R}\right) - \overline{P}_1(1-\nu_2^2)\cos\left(\nu_2 \sqrt{(\mathrm{Pd}')}\,\frac{x}{R}\right)}{\overline{Q}_2\,\overline{P}_1 - \overline{Q}_1\,\overline{P}_2} \times$$

$$\times \exp\left(-\mathrm{Pd}'\mathrm{Fo}\right) - \sum_{n=1}^{\infty} \left[C_{n1}^{*}(1-\nu_2^2)\cos \nu_2 \mu_n \frac{x}{R} -\right.$$

$$\left. - C_{n2}^{*}(1-\nu_1^2)\cos \nu_1 \mu_n \frac{x}{R}\right] \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (3.2)$$

where

$$C_{nj} = 2\,\frac{\varepsilon\,\mathrm{Ko}\,Q_{nj} - \left[\varepsilon\,\mathrm{Ko}\,K_1 + W_0 - \dfrac{W_c}{1 - \mathrm{Pd}'/\mu_n^2}\right] P_{nj}}{\mu_n \nu_n},$$

$$C_{nj}^{*} = \frac{C_{nj}}{\varepsilon\,\mathrm{Ko}}, \qquad (3.3)$$

$$\overline{Q}_j = [1 + K_1(1-\nu_j^2)]\cos \nu_j \sqrt{(\mathrm{Pd}')} -$$

$$- \frac{1}{\mathrm{Bi}_q} \nu_j \sqrt{(\mathrm{Pd}')} \sin \nu_j \sqrt{(\mathrm{Pd}')},$$

$$\overline{P}_j = (1-\nu_j^2)\cos \nu_j \sqrt{(\mathrm{Pd}')} - [(1-\nu_j^2) + \mathrm{Fc}] \times$$

$$\times \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_m} \sin \nu_j \sqrt{\mathrm{Pd}'}.$$

The mean non-dimensional transfer potentials are

$$\overline{T} = - W_0 + W_c N \exp\left(-\mathrm{Pd}'\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} D_n \exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (3.4)$$

$$\overline{\Theta} = 1 + W_c N^* \exp(-\mathrm{Pd'Fo}) - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}). \tag{3.5}$$

Here

$$N = \frac{\overline{\overline{P}}_1 \frac{\sin(\nu_2 \sqrt{\mathrm{Pd'}})}{\nu_2 \sqrt{\mathrm{Pd'}}} - \overline{\overline{P}}_2 \frac{\sin(\nu_1 \sqrt{\mathrm{Pd'}})}{\nu_1 \sqrt{\mathrm{Po'}}}}{\overline{\overline{Q}}_2 \overline{\overline{P}}_1 - \overline{\overline{Q}}_1 \overline{\overline{P}}_2},$$

$$N^* = \frac{1}{\varepsilon \mathrm{Ko}} \times$$

$$\times \frac{\overline{\overline{P}}_2(1-\nu_1^2) \frac{\sin(\nu_1 \sqrt{\mathrm{Pd'}})}{\nu_1 \sqrt{\mathrm{Pd'}}} - \overline{\overline{P}}_1(1-\nu_2^2) \frac{\sin(\nu_2 \sqrt{\mathrm{Pd'}})}{\nu_2 \sqrt{\mathrm{Pd'}}}}{\overline{\overline{Q}}_2 \overline{\overline{P}}_1 - \overline{\overline{Q}}_1 \overline{\overline{P}}_2}.$$

The form of the expressions for D_n and D_n^* is as usual. The local and mean non-dimensional rates of change of the potentials are

$$\frac{dT}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - \right.$$

$$\left. - C_{n1} \cos \nu_2 \mu_n \frac{x}{R} \right) \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}) - W_c \mathrm{Pd'} \times$$

$$\times \frac{\overline{\overline{P}}_1 \cos\left(\nu_2 \sqrt{(\mathrm{Pd'})} \frac{x}{R}\right) - \overline{\overline{P}}_2 \cos\left(\nu_1 \sqrt{(\mathrm{Pd'})} \frac{x}{R}\right)}{\overline{\overline{Q}}_2 \overline{\overline{P}}_1 - \overline{\overline{Q}}_1 \overline{\overline{P}}_2} \exp(-\mathrm{Pd'Fo}), \tag{3.6}$$

$$\frac{d\overline{T}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}) - W_c \mathrm{Pd'} N, \tag{3.7}$$

$$\frac{d\,\Theta}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \left[C_{n1}^*(1-\nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} - \right.$$

$$\left. - C_{n2}^*(1-\nu_1^2) \cos \nu_1 \mu_n \frac{x}{R} \right] \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}) - \frac{W_c}{\varepsilon \mathrm{Ko}} \times$$

$$\times \mathrm{Pd'} \frac{\overline{\overline{P}}_2(1-\nu_1^2) \cos\left(\nu_1 \sqrt{(\mathrm{Pd'})} \frac{x}{R}\right) - \overline{\overline{P}}_1(1-\nu_2^2) \cos\left(\nu_2 \sqrt{(\mathrm{Pd'})} \frac{x}{R}\right)}{Q_2 \overline{\overline{P}}_1 - \overline{Q}_1 \overline{\overline{P}}_2} \times$$

$$\times \exp(-\mathrm{Pd'\,Fo}), \tag{3.8}$$

$$\frac{d\overline{\Theta}}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^* \mu_n^2 \exp\left(-\mu_n^2\,\mathrm{Fo}\right) - W_c\,\mathrm{Pd}'\,N^*. \qquad (3.9)$$

In the solutions (3.1), (3.2), (3.4) and (3.9) $\mathrm{Pd}' = \varkappa R^2/a_q$ is the Predvoditelev criterion for problems where the temperature of the medium changes exponentially with the time; $w_a = t_a^0/(t_a^0 - t_0)$, $w_0 = t_0/t_a^0 - t_0$ are the parametrical criteria for the temperature.

(b) *Solution for the sphere. Constant initial conditions.*

$$T = \frac{t(r,\tau) - t_0}{t_c^0 - t_0} = -W_0 - W_c \times$$

$$\times \frac{\bar{P}_1 \dfrac{\sin\left(r_2 V(\mathrm{Pd}')\dfrac{r}{R}\right)}{\dfrac{r}{R}} - \bar{P}_2 \dfrac{\sin\left(r_1 V(\mathrm{Pd}')\dfrac{r}{R}\right)}{\dfrac{r}{R}}}{\bar{Q}_2 \bar{P}_2 - \bar{Q}_1 P_2} \exp\left(-\mathrm{Pd}'\,\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin r_2 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} - C_{n2}\frac{\sin r_1 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}}\right)\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (3.10)$$

$$\Theta = \frac{\theta_0 - \theta(r,\tau)}{\theta_0 - \theta_p} = 1 + \frac{W_c}{\varepsilon\,\mathrm{Ko}}\left(\bar{Q}_2\bar{P}_2 - \bar{Q}_1\bar{P}_2\right)^{-1} \times$$

$$\times\left[\bar{P}_2(1 - r_1^2)\frac{\sin\left(r_1 V(\mathrm{Pd}')\dfrac{r}{R}\right)}{\dfrac{r}{R}} - \right.$$

$$\left. - P_1(1 - r_2^2)\frac{\sin\left(r_2 V(\mathrm{Pd}')\dfrac{r}{R}\right)}{\dfrac{r}{R}}\right]\exp\left(-\mathrm{Pd}'\,\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty}\left[C_{n2}^*(1 - r_1^2)\frac{\sin r_1 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} - \right.$$

$$\left. - C_{n1}^*(1 - r_2^2)\frac{\sin r_2 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}}\right]\exp\left(-\mu_n^2\,\mathrm{Fo}\right). \qquad (3.11)$$

C_{nj} and C^*_{nj} are determined by equation (3.3), other terms were determined as in the preceding solutions for the sphere, and

$$\bar{\bar{Q}}_j = \frac{1}{\mathrm{Bi}_q} \nu_j \sqrt{(\mathrm{Pd}')} \cos(\nu_j \sqrt{\mathrm{Pd}'}) + \\ + \left[1 - \frac{1}{\mathrm{Bi}_a} + K_1(1 - \nu_j^2)\right] \sin(\nu_j \sqrt{\mathrm{Pd}'}),$$

$$P_j = \left[(1 - \nu_j^2) + \mathrm{Fe}\right] \frac{1}{\mathrm{Bi}_m} \nu_j \sqrt{(\mathrm{Pd}')} \cos \nu_j \sqrt{\mathrm{Pd}'} + \\ + \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m}\right] \sin(\nu_j \sqrt{\mathrm{Pd}'}).$$

The characteristic roots μ_n, as usual are determined by the equation

$$Q_{n2} P_{n1} - Q_{n1} P_{n2} = 0.$$

The mean non-dimensional temperature and mass potential are determined by the equations:

$$\bar{T} = -W_0 + W_c N \exp(-\mathrm{Pd}'\,\mathrm{Fo}) - \\ - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (3.12)$$

$$\bar{\Theta} = 1 + W_c N^* \exp(+\mathrm{Pd}'\,\mathrm{Fo}) - \\ - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (3.13)$$

where

$$N = \frac{\bar{\bar{P}}_1 \xi_2 - \bar{\bar{P}}_2 \xi_1}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2},$$

$$N^* = \frac{1}{\varepsilon\,\mathrm{Ko}} \frac{(1 - \nu_1^2)\,\xi_1 \bar{\bar{P}}_2 - (1 - \nu_2^2)\,\xi_2 \bar{\bar{P}}_1}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2},$$

$$\xi_j = \frac{3\left[\sin(\nu_j \sqrt{\mathrm{Pd}'}) - \nu_j \sqrt{(\mathrm{Pd}')} \cos(\nu_j \sqrt{\mathrm{Pd}'})\right]}{(\nu_j \sqrt{\mathrm{Pd}'})^2},$$

$$D_n = D_{n1} - D_{n2},\ D_n^* = D_{n2}^* - D_{n1}^*.$$

Differentiating equations (3.10) – (3.13) with respect to the non-dimensional time, it is easy to obtain solutions for the rate of heating of the moist material and for the rate of drying.

(c) *Solutions for an infinite plate. Parabolic initial conditions.*

$$T_1 = \frac{t(x,\tau) - t_n}{t_c^0 - t^0} = -W_{nn} +$$

$$+ W_{cn} \frac{\overline{P}_1 \cos\left(\nu_2 V(\mathrm{Pd}')\frac{x}{R}\right) - \overline{P}_2 \cos\left(\nu_1 V(\mathrm{Pd}')\frac{x}{R}\right)}{\overline{Q}_2 \overline{P}_1 - \overline{Q}_1 \overline{P}_2} \times$$

$$\times \exp(-\mathrm{Pd}'\,\mathrm{Fo}) - \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - \right.$$

$$\left. - C_{n1} \cos \nu_2 \mu_n \frac{x}{R}\right) \exp(-\mu_n^2 \mathrm{Fo}), \qquad (3.14)$$

$$\Theta_1 = \frac{\theta_n - \theta(x,\tau)}{\theta_n - \theta_p} = 1 + \frac{W_{cn}}{\varepsilon\,\mathrm{Ko}} \times$$

$$\times \left[\overline{P}_2(1-\nu_1^2)\cos\left(\nu_1 V(\mathrm{Pd}')\frac{x}{R}\right) - \right.$$

$$\left. - P_1(1-\nu_2^2)\cos\left(\nu_2 V(\mathrm{Pd}')\frac{x}{R}\right)\right] \frac{\exp(-\mathrm{Pd}'\,\mathrm{Fo})}{\overline{Q}_2 \overline{P}_1 - \overline{Q}_1 \overline{P}_2} -$$

$$- \sum_{n=1}^{\infty} \left[C_{n1}^*(1-\nu_2^2)\cos \nu_2 \mu_n \frac{x}{R} - C_{n2}^*(1-\nu_1^2)\cos \nu_1 \mu_n \frac{x}{R}\right] \times$$

$$\times \exp(-\mu_n^2 \mathrm{Fo}), \qquad (3.15)$$

where

$$C_{nj} = 2\,\frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \nu_n}, \; C_{nj}^* = \frac{C_{nj}}{\varepsilon\,\mathrm{Ko}}, \qquad (3.16)$$

$$G_1 = 1 - 2W\left(\frac{1}{\mathrm{Bi}_q} - \frac{1}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,K_1 -$$

$$- 2\,\varepsilon\,\mathrm{Ko}\,K_2 \frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2} - W_{cn} + \frac{W_{cn}}{1 - \mathrm{Pd}'/\mu_n^2},$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}.$$

The expressions for the mean values will be the same as in the case of constant initial conditions — equations (3.4) and (3.5). Here, however, we must take into account that in place of W_0 we have W_{nn}, and in place of W_c, W_{cn}. Differences will be introduced in D_n and D_n^*, or, more exactly, in C_{nj} and C_{nj}^*. The latter must be determined by equation (3.16):

$$W_{nn} = W_{cn} - 1 = \frac{t_n}{t_c^0 - t_n}.$$

(d) *Solutions for the sphere. Parabolic initial conditions.*

$$T_1 = \frac{t(r,\tau) - t_n}{t_c^0 - t_n} = -W_{nn} +$$

$$+ W_{cn} \frac{\overline{\overline{P}}_1 \frac{\sin[\nu_2]'(\mathrm{Pd}')\, r/R]}{r/R} - \overline{\overline{P}}_2 \frac{\sin[\nu_1]'(\mathrm{Pd}')\, r/R]}{r/R}}{\overline{\overline{Q}}_2 \overline{\overline{P}}_1 - \overline{\overline{Q}}_1 \overline{\overline{P}}_2} \times$$

$$\times \exp(-\mathrm{Pd}'\,\mathrm{Fo}) - \sum_{n=1}^{\infty} \left(C_{n1} \frac{\sin \nu_2 \mu_n\, r/R}{r/R} - \right.$$

$$\left. C_{n2} \frac{\sin \nu_1 \mu_n\, r/R}{r/R} \right) \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (3.17)$$

$$\Theta_1 = \frac{\theta_n - \theta(r,\tau)}{\theta_n - \theta_p} = 1 + \frac{W_{cn}}{\varepsilon\,\mathrm{Ko}} \times$$

$$\times \left[\overline{\overline{P}}_2 (1 - \nu_1^2) \frac{\sin[\nu_1]'(\mathrm{Pd}')\, r/R]}{r/R} - \right.$$

$$\left. - \overline{\overline{P}}_1 (1 - \nu_2^2) \frac{\sin[\nu_2]'(\mathrm{Pd}')\, r/R]}{r/R} \right] \frac{\exp(-\mathrm{Pd}'\mathrm{Fo})}{\overline{\overline{Q}}_2 \overline{\overline{P}}_1 - \overline{\overline{Q}}_1 \overline{\overline{P}}_2} -$$

$$- \sum_{n=1}^{\infty} \left[C_{n2}^* (1 - \nu_1^2) \frac{\sin \nu_1 \mu_n\, r/R}{r/R} - \right.$$

$$\left. - C_{n1}^* (1 - \nu_2^2) \frac{\sin \nu_2 \mu_n\, r/R}{r/R} \right] \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (3.18)$$

where

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \quad C_{nj}^* = \frac{C_{nj}}{\varepsilon\,\mathrm{Ko}},$$

$$G_1 = 1 - 2W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,K_1 -$$

$$- G\,\varepsilon\,\mathrm{Ko}\,K_2\,\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2} -$$

$$- W_{cn} + \frac{W_{cn}}{1 - \mathrm{Pd}'/\mu_n^2}\,,$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right)\varepsilon\,\mathrm{Ko}\,.$$

§ 4. Generalized Solutions of the System of Equations for Heat and Mass Transfer

If we compare the different solutions of the problems in §§ 1–3 for bodies of the same form, as well as solutions for cylinders not dealt with in this monograph, we find that they contain numerous common elements (for example, ν_n, Q_{nj}, P_{nj}, A_{nj}, B_{nj}, the characteristic roots, etc.). This makes it possible to present final solutions in a form applicable both to problems with various initial conditions and to those with different modes of change of the temperature of the external medium. We give below such general solutions for the infinite plate, the sphere and infinite cylinder.

The generalized solution for heat and mass transfer in an infinite plate may be presented in the following form:

$$T = 1 + \Pi - \sum_{n=1}^{\infty}\,(C_{n2}\cos\nu_1\mu_n x/R - C_{n1}\cos\nu_2\mu_n x/R)\times$$

$$\times\exp(-\mu_n^2\,\mathrm{Fo})\,,$$

$$\Theta = 1 + \Pi^* - \sum_{n=1}^{\infty}\,[C_{n1}^*(1-\nu_2^2)\cos\nu_2\mu_n x/R -$$

$$- C_{n2}^*(1-\nu_1^2)\cos\nu_1\mu_n x/R]\exp(-\mu_n^2\,\mathrm{Fo})\,,$$

where

(a) with constant initial conditions

$$C_{nj} = 2\,\frac{P_{nj} + \varepsilon\,\mathrm{Ko}\,(Q_{nj} - K_1 P_{nj}) + MP_{nj}}{\mu_n\nu_n}\,,$$

$$C^*_{nj} = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(x, 0) = t_0, \ \theta(x, 0) = \theta_0;$$

(b) with parabolic initial conditions

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \varphi_n}, \quad C^*_{nj} = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(x, 0) = t_n \ \theta(x, 0) = \theta_n.$$

Here

$$\nu_j^2 = \frac{1}{2}\left\{\left(1 + \varepsilon \mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right) + (-1)^j \times \right.$$

$$\left. \times \sqrt{\left[\left(1 + \varepsilon \mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}\right]}\right\},$$

$$\varphi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n +$$

$$+ \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n +$$

$$+ \frac{(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\,\mathrm{Pn}}{\mathrm{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n),$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n,$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - [(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\,\mathrm{Pn}] \times$$

$$\times \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n,$$

$$G_1 = 1 - 2W \left(\frac{1}{\mathrm{Bi}_q} - \frac{1}{\mu_n^2}\right) - \varepsilon \mathrm{Ko}\, K_1 -$$

$$- 2 \varepsilon \mathrm{Ko}\, K_2 \frac{(K_1 + 1) \mathrm{Lu}}{\mu_n^2} + M,$$

$$G_2 = 2 \varepsilon \mathrm{Ko}\, K_2 \left(\frac{1}{\mathrm{Bi}_m} - \frac{\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon \mathrm{Ko},$$

μ_n is determined from the solutions of the equation $Q_2 P_1 = Q_1 P_2$.

For $t_c = \text{const}$.

$$\Pi = 0, \quad \Pi^* = 0, \quad M = 0.$$

For $t_c = t_c^0 - b\tau$

$$\Pi = -\,\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right],$$

$$\Pi^* = \frac{1}{2}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{x^2}{R^2}\right), \quad M = \frac{\mathrm{Pd}}{\mu_n^2},$$

For $t_a = t_a^0 e^{-\varkappa\tau}$

$$\Pi = -\,W_c + W_c \frac{\overline{P}_1 \cos\left(r_2 V(\mathrm{Pd}')\frac{x}{R}\right) - \overline{P}_2 \cos\left(r_1 V(\mathrm{Pd}')\frac{x}{R}\right)}{\overline{Q}_2 \overline{P}_1 - \overline{Q}_1 \overline{P}_2} \times$$

$$\times \exp\left(-\,\mathrm{Pd}'\mathrm{Fo}\right),$$

$$\Pi^* = \frac{W_c}{\varepsilon\,\mathrm{Ko}} \times$$

$$\times \frac{\overline{P}_2(1 - r_1^2)\cos\left(r_1 V(\mathrm{Pd}')\frac{x}{R}\right) - \overline{P}_1(1 - r_2^2)\cos\left(r_2 V(\mathrm{Pd}')\frac{x}{R}\right)}{\overline{Q}_2 \overline{P}_1 - \overline{Q}_1 \overline{P}_2} \times$$

$$\times \exp\left(-\,\mathrm{Pd}'\mathrm{Fo}\right),$$

$$M = -\,W_c + \frac{W_c}{1 - \dfrac{\mathrm{Pd}'}{\mu_n^2}},$$

$$\overline{Q}_j = [1 + K_1(1 - r_j^2)]\cos\left(r_j V\mathrm{Pd}'\right) -$$

$$- \frac{r_j V\mathrm{Pd}'}{\mathrm{Bi}_q}\sin\left(r_j V\mathrm{Pd}'\right),$$

$$\overline{P}_j = (1 - r_j^2)\cos\left(r_j V\mathrm{Pd}'\right) -$$

$$- [(1 - r_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}]\, r_j \frac{V\mathrm{Pd}'}{\mathrm{Bi}_m}\sin\left(r_j V\mathrm{Pd}'\right).$$

The generalized solution for the internal heat and mass transfer in the sphere are written:

$$T = 1 + \Pi - \sum_{n=1}^{\infty} \left(C_{n1} \frac{\sin \nu_2 \mu_n \frac{r}{R}}{\frac{r}{R}} - \right.$$

$$\left. - C_{n2} \frac{\sin \nu_1 \mu_n \frac{r}{R}}{\frac{r}{R}} \right) \exp\left(-\mu_n^2 \mathrm{Fo}\right),$$

$$\Theta = 1 + \Pi^* - \sum_{n=1}^{\infty} \left[C_{n2}^*(1 - \nu_1^2) \frac{\sin \nu_1 \mu_n \frac{r}{R}}{\frac{r}{R}} - \right.$$

$$\left. - C_{n1}^*(1 - \nu_2^2) \frac{\sin \nu_2 \mu_n \frac{r}{R}}{\frac{r}{R}} \right] \exp\left(-\mu_n^2 \mathrm{Fo}\right),$$

where (a) has constant initial conditions

$$C_{nj} = 2 \frac{P_{nj} + \varepsilon \mathrm{Ko}\,(Q_{nj} - K_1 P_{nj}) + M P_{nj}}{\mu_n \psi_n}, \; C_{nj}^* = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(r, 0) = t_0, \; \theta(r, 0) = \theta_0;$$

(b) has parabolic initial conditions

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \; C_{nj}^* = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(r, 0) = t_n, \; \theta(r. 0) = \theta_n.$$

Here

$$\nu_j^2 = \frac{1}{2} \left\{ \left(1 + \varepsilon \mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left[\left(1 + \varepsilon \mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}\right]} \right\},$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = [1 + K_1(1 - \nu_j^2)] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2)\cos\nu_j\mu_n - [(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\sin\nu_j\mu_n,$$

$$Q_{nj} = \left[1 - \frac{1}{\mathrm{Bi}_q} + K_1(1 - \nu_j^2)\right]\sin\nu_j\mu_n + \frac{\nu_j\mu_n}{\mathrm{Bi}_q}\cos\nu_j\mu_n,$$

$$P_{nj} = \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}}{\mathrm{Bi}_m}\right]\sin\nu_j\mu_n +$$
$$+ [(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\cos\nu_j\mu_n,$$

$$G_1 = 1 - 2\,W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}\,K_1 -$$
$$- 6\,\varepsilon\,\mathrm{Ko}\,K_2\frac{(K_1 + 1)\,\mathrm{Lu}}{\mu_n^2} + M,$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko},$$

μ_n is defined by the solution of the equation

$$Q_2 P_1 = Q_1 P_2.$$

For $t_c = \mathrm{const.}$

$$\Pi = 0,\ \Pi^* = 0,\ M = 0,$$

For $t_c = t_c^0 - b\tau$

$$\Pi = -\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right],$$

$$\Pi^* = \frac{1}{6}\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right),$$

$$M = \frac{\mathrm{Pd}}{\mu_n^2}.$$

For $t_c = t_c^0 e^{-\varkappa\tau}$

$$\Pi = -W_c + W_c\frac{\overline{P}_1\dfrac{\sin\left(r_2\sqrt{(\mathrm{Pd}')}\dfrac{r}{R}\right)}{\dfrac{r}{R}} - \overline{P}_2\dfrac{\sin\left(r_1\sqrt{(\mathrm{Pd}')}\dfrac{r}{R}\right)}{\dfrac{r}{R}}}{\overline{Q}_2\overline{P}_1 - \overline{Q}_1\overline{P}_2} \times$$

$$\times \exp(-\mathrm{Pd}'\mathrm{Fo}),$$

$$\Pi^* = \frac{W_c}{\varepsilon \mathrm{Ko}} \left[\overline{P}_2 (1 - \nu_1^2) \frac{\sin\left(\nu_1 \sqrt{(\mathrm{Pd}')} \frac{r}{R}\right)}{\frac{r}{R}} - \right.$$

$$\left. - \overline{P} (1 - \nu_2^2) \frac{\sin\left(\nu_2 \sqrt{(\mathrm{Pd}')} \frac{r}{R}\right)}{\frac{r}{R}} \right] \frac{\exp(-\mathrm{Pd}'\mathrm{Fo})}{\overline{\overline{Q}}_2 \overline{P}_1 - \overline{\overline{Q}}_1 \overline{P}_2},$$

$$M = - W_c + \frac{W_c}{1 - \frac{\mathrm{Pd}'}{\mu_n^2}},$$

$$\overline{\overline{Q}}_j = \left[1 - \frac{1}{\mathrm{Bi}_q} + K_1 (1 - \nu_j^2) \right] \sin(\nu_j \sqrt{\mathrm{Pd}'}) -$$

$$- \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_q} \cos(\nu_j \sqrt{\mathrm{Pd}'}),$$

$$\overline{P}_j = \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\, \mathrm{Pn}}{\mathrm{Bi}_m} \right] \sin(\nu_j \sqrt{\mathrm{Pd}'}) +$$

$$+ \left[(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\, \mathrm{Pn} \right] \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_m} \cos(\nu_j \sqrt{\mathrm{Pd}'}).$$

The generalized solution for heat and mass transfer in a infinite cylinder may be presented in the following form

$$T = 1 + \Pi - \sum_{n=1}^{\infty} \left[C_{n2} J_0\left(\nu_1 \mu_n \frac{r}{R}\right) - \right.$$

$$\left. - C_{n1} J_0\left(\nu_2 \mu_n \frac{r}{R}\right) \right] \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\Theta = 1 + \Pi^* - \sum_{n=1}^{\infty} \left[C_{n1}^* (1 - \nu_2^2) J_0(\nu_2 \mu_n r/R) - \right.$$

$$\left. - C_{n2}^* (1 - \nu_1^2) J_0(\nu_1 \mu_n r/R) \right] \exp(-\mu_n^2 \mathrm{Fo}),$$

where

(a) with constant initial conditions

$$C_{nj} = 2 \frac{P_{nj} + \varepsilon \mathrm{Ko}\, (Q_{nj} - K_1 P_{nj}) + M P_{nj}}{\mu_n \Psi_n},$$

$$C_{nj}^{*} + \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(\tau, 0) = t_0, \; \Theta(\tau, 0) = \Theta_0;$$

(b) with parabolic initial conditions

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \varphi_n}, \; C_{nj}^{*} = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t(\tau, 0) = t_w, \; \Theta(\tau, 4) = \Theta_w.$$

Here

$$\nu_j^2 = \frac{1}{2} \left\{ \left(1 + 2 \,\mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left[\left(1 + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + \frac{1}{\mathrm{Lu}}\right)^2 - 4/\mathrm{Lu}\right]} \right\},$$

$$\varphi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$B_{nj} = (1 - \nu_j^2) J_1(\nu_j \mu_n) + \frac{(1 - \nu_j^2) + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn}}{\mathrm{Bi}_m} \nu_j \mu_n J_0(\nu_j \mu_n),$$

$$A_{nj} = [1 + K_1(1 - \nu_j^2)] J_1(\nu_j \mu_n) + \frac{1}{\mathrm{Bi}_q} \nu_j \mu_n J_0(\nu_j \mu_n),$$

$$Q_{nj} = [1 + K_1(1 - \nu_j^2)] J_0(\nu_j \mu_n) - \frac{1}{\mathrm{Bi}_q} \nu_j \mu_n J_1(\nu_j \mu_n),$$

$$P_{nj} = (1 - \nu_j^2) J_0(\nu_j \mu_n) - \frac{(1 - \nu_j^2) + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn}}{\mathrm{Bi}_m} \nu_j \mu_n J_1(\nu_j \mu_n),$$

$$G_1 = 1 - 2 W \left(\frac{1}{\mathrm{Bi}_q} - \frac{2}{\mu_n^2}\right) -$$

$$- \varepsilon \,\mathrm{Ko}\, K_1 - 4 \varepsilon \,\mathrm{Ko}\, K_2 \frac{(K_1 + 1)\,\mathrm{Lu}}{\mu_n^2} - M,$$

$$G_2 = 2 \varepsilon \,\mathrm{Ko}\, K_2 \left(\frac{1}{\mathrm{Bi}_m} - \frac{2\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon \,\mathrm{Ko},$$

μ_n is determined from the solutions of the equation

$$Q_2 P_1 = Q_1 P_2,$$

J_0 and J_1 are Bessel functions of the first kind of zero and first orders.

For $t_c = \text{const.}$

$$\Pi = 0,\ \Pi^* = 0,\ M = 0$$

For $t_c = t_c^0 - b\tau$

$$\Pi = -\,\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{4}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right],$$

$$\Pi^* = \frac{1}{4}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right),$$

$$M = \frac{\mathrm{Pd}}{\mu_n^2}.$$

For $t_c = t_c^0\, e^{-x\tau}$

$$\Pi = -\,W_c +$$

$$+\,W_c\,\frac{\overline{\overline{P}}_1 I_0\left[\nu_2 \sqrt{(\mathrm{Pd}')}\, r/R\right] - \overline{\overline{P}}_2 I_0\left[\nu_1 \sqrt{(\mathrm{Pd}')}\, r/R\right]}{\overline{\overline{Q}}_2\,\overline{\overline{P}}_1 - \overline{\overline{Q}}_1\,\overline{\overline{P}}_2}\exp\left(-\,\mathrm{Pd}'\,\mathrm{Fo}\right),$$

$$\Pi^* = \frac{W_c}{\varepsilon\,\mathrm{Ko}} \times \frac{\overline{\overline{P}}_2(1-\nu_1^2)\, I_0\left[\nu_1 \sqrt{(\mathrm{Pd})}\, r/R\right]}{\overline{\overline{Q}}_2\,\overline{\overline{P}}_1 - \overline{\overline{Q}}_1\,\overline{\overline{P}}_2} -$$

$$-\,\frac{\overline{\overline{P}}_1(1-\nu_2^2)\, I_0\left[\nu_2 \sqrt{(\mathrm{Pd}')}\, r/R\right]}{\overline{\overline{Q}}_2\,\overline{\overline{P}}_1 - \overline{\overline{Q}}_1\,\overline{\overline{P}}_2}\exp\left(-\,\mathrm{Pd}'\,\mathrm{Fo}\right),$$

$$M = -\,W_c + \frac{W_c}{1 - \dfrac{\mathrm{Pd}'}{\mu_n^2}},$$

$$\overline{\overline{Q}}_j = \left[1 + K_1(1-\nu_j^2)\right] I_0\left[\nu_j \sqrt{(\mathrm{Pd}')}\right] - \frac{1}{\mathrm{Bi}_q}\,\nu_j \sqrt{(\mathrm{Pd}')}\, I_1\left[\nu_j \sqrt{(\mathrm{Pd}')}\right],$$

$$\overline{\overline{P}}_j = (1-\nu_j^2)\, I_0\left[\nu_j \sqrt{(\mathrm{Pd}')}\right] -$$

$$-\,\frac{(1-\nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}}{\mathrm{Bi}_m}\,\nu_j \sqrt{(\mathrm{Pd}')}\, I_1\left[\nu_j \sqrt{(\mathrm{Pd}')}\right].$$

§ 5. Special Cases of Heat and Mass Transfer. Certain Questions of Limiting Transitions in the System of Equations of Heat and Mass Transfer

The non-stationary fields of transfer potential for heat and mass are, in special cases of heat and mass transfer, best obtained from the solutions of the complete system of equations (I – VI) by passing to the limit. However, the direct application of limiting methods to the solution of this

system of equations gives a result which does not coincide with that obtained from the incomplete system of equations ($\delta = 0$, $\varepsilon = 0$, etc.). Investigation shows that to obtain correct results by limiting methods, an important consideration is the form of the characteristic equation after passing to the limit, and the "form" of the roots thus obtained. Thus for an infinite plate the characteristic equation (1.20) has, as a result of δ-transition (and also of Pn-transition and Ko-transition) the form

$$\left(\cos\mu - \frac{1}{\mathrm{Bi}_q}\mu\sin\mu\right) \times$$

$$\times \left(\cos\frac{\mu}{\sqrt{\mathrm{Lu}}} - \frac{1}{\mathrm{Bi}_m}\frac{\mu}{\sqrt{\mathrm{Lu}}}\sin\frac{\mu}{\sqrt{\mathrm{Lu}}}\right) = 0\,, \qquad (5.1')$$

i.e. the old single form of the root μ_{n0}, contained in the solution of the complete system of equations, takes a double form: μ and $\mu^* = \mu/\sqrt{\mathrm{Lu}}$. Passage to the limit in the system of equations of heat and mass transfer will be correct if we repeat the initial infinite sum of the general solution as many times as there are forms of the roots in the transformed characteristic equation. For example, the infinite sum of equation (1.28) must be repeated twice, substituting instead of μ_{n0} first of all μ and then μ^*, since in the characteristic equation (5.1') we have two "forms" of the root. Thereafter the transition can be performed in the usual way. Passing to the limit permits us to verify the correctness of the solutions of the complete system of equations, to obtain easily and quickly the solutions of incomplete systems on the basis of one general solution, and finally to disclose new and interesting special features of the process. The working out of methods of proceeding to the limit for the system of equations of heat and mass transfer has revealed a new physical significance for the criterion $\sqrt{\mathrm{Lu}}$ as a measure of the relation between heat exchange and mass exchange, and has shown the possibility, from the formal mathematical point of view, of carrying out calculations of heat and mass transfer in incomplete

systems ($\delta = 0$, $\varepsilon = 0$) in terms of either mass exchange quantities, or heat exchange quantities. It is interesting to note that $\sqrt{\text{Lu}}$ fulfils in processes of heat and mass transfer the same role that the square of the velocity of light c^2 fulfils in the well-known Einstein equation: $E = mc^2$, determining the interrelationship between energy and mass quantities.

By the method which has been explained of passing to the limit we can obtain solutions for special cases of heat and mass transfer in moist bodies.

(a) *Absence of thermal transfer of material in the body* ($\delta = 0$, $\text{Pn} = 0$). The general solution for heat and mass transfer in an infinite plate with constant initial conditions may be written in the following way:

$$T = \frac{t(x,\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} A_n \left(1 + \frac{\varphi_n}{\psi_n}\right) \cos \mu_n \frac{x}{R} \times$$

$$\times \exp\left(-\mu_n^2 \text{Fo}\right) - \sum_{n=1}^{\infty} A_n^* \text{Ko} \left(\frac{\varepsilon \text{Lu}}{1 - \text{Lu}} \cos \mu_n^* \frac{x}{R} + \right.$$

$$\left. + \frac{\varphi_n}{\psi_n^*} \cos \sqrt{(\text{Lu})}\, \mu_n^* \frac{x}{R}\right) \exp\left(-\mu_n^{*2} \text{Fo}^*\right),$$

$$\Theta = \frac{\theta_0 - \theta(x,\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} A_n^* \cos \mu_n^* \frac{x}{R} \exp\left(-\mu_n^{*2} \text{Fo}^*\right), \quad (5.1)$$

where

$$\varphi_n = \text{Ko}\left[\frac{\text{Bi}_m}{\text{Bi}_q}\left(1 - \varepsilon - \frac{\varepsilon}{\text{Lu} - 1}\right)\sqrt{(\text{Lu})}\, \mu_n \sin \frac{\mu_n}{\sqrt{\text{Lu}}} + \right.$$

$$\left. + \frac{\varepsilon \text{Lu}\, \text{Bi}_m}{\text{Lu} - 1} \cos \frac{\mu_n}{\sqrt{\text{Lu}}}\right],$$

$$\varphi_n^* = \text{Lu}\left[\left(1 - \varepsilon - \frac{\varepsilon}{\text{Lu} - 1}\right)\mu_n^* \sin \mu_n^* + \frac{\varepsilon\, \text{Bi}_q}{\text{Lu} - 1} \cos \mu_n^*\right],$$

$$\psi_n = \text{Bi}_m \cos \frac{\mu_n}{\sqrt{\text{Lu}}} - \frac{\mu_n}{\sqrt{\text{Lu}}} \sin \frac{\mu_n}{\sqrt{\text{Lu}}},$$

$$\psi_n^* = \text{Bi}_q \cos \sqrt{(\text{Lu})}\, \mu_n^* - \sqrt{(\text{Lu})}\, \mu_n^* \sin \sqrt{(\text{Lu})}\, \mu_n^*,$$

μ_n and μ_n^* are the roots of the characteristic equations,

$$\cot \mu = \frac{1}{\text{Bi}_q} \mu \quad \text{and} \quad \cot \mu^* = \frac{1}{\text{Bi}_m} \mu^*, \quad (5.2)$$

and the coefficients A_n and A_n^* are defined by the expressions

$$A_n = \frac{2 \sin \mu_n}{\mu_n + \sin \mu_n \cos \mu_n}, \quad A_n^* = \frac{2 \sin \mu_n^*}{\mu_n^* + \sin \mu_n^* \cos \mu_n^*}. \quad (5.3)$$

The values of the characteristic roots are given in Table 23, and the values of the coefficients A_n and A_n^* in Table 24. The same quantities are represented in Figs. 17 and 18.

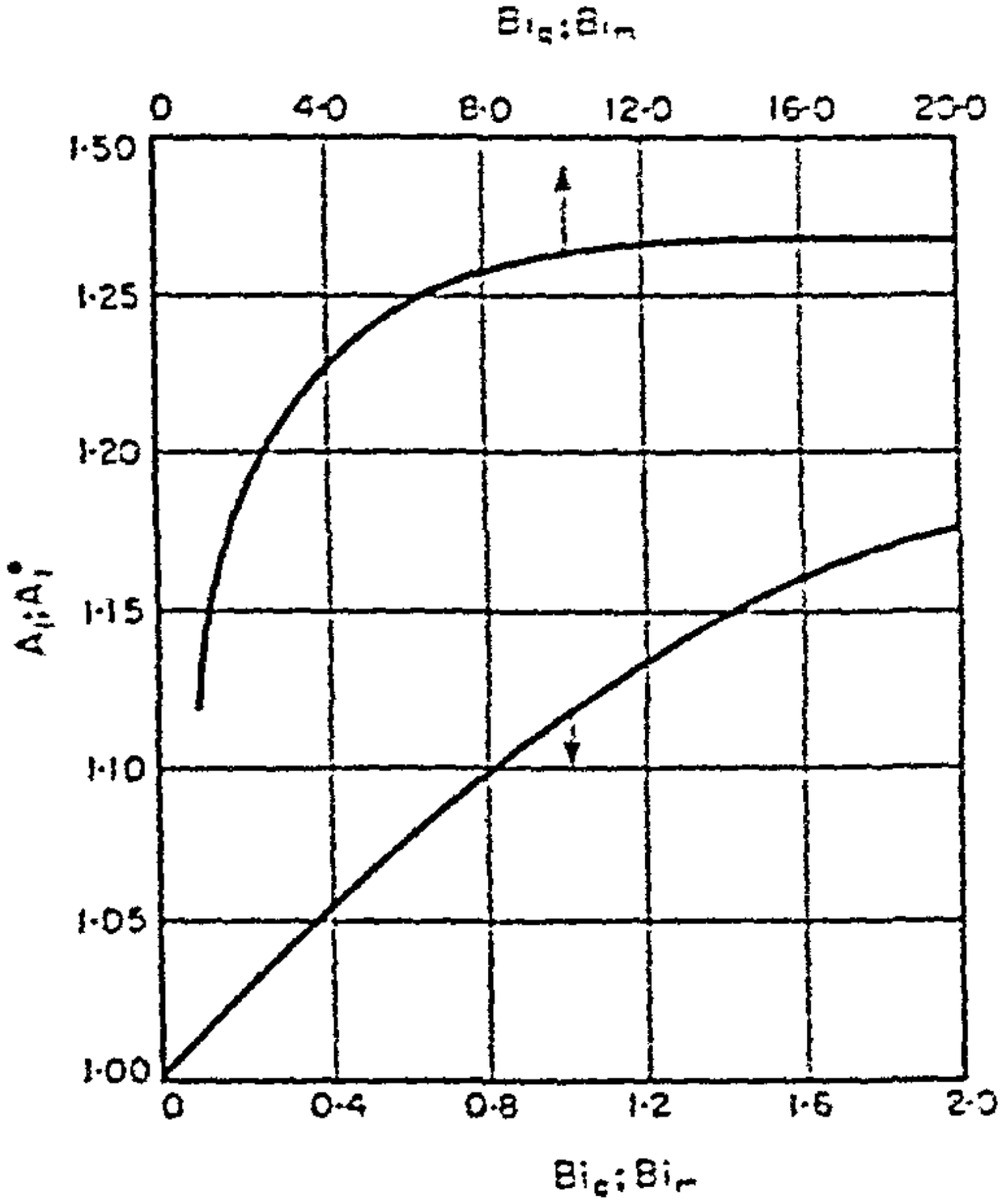


FIG. 18. Relation between A_1, A_1^* and the Biot criteria for a plate

The general solution for heat and mass transfer in a sphere with constant initial conditions may be written:

$$T = \frac{t(r, \tau) - t_0}{t_c - t_0} = 1 -$$

$$- \sum_{n=1}^{\infty} A_n \left(1 + \frac{\mu_n}{\nu_n}\right) \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left(-\mu_n^2 \mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} A_n^* \operatorname{Ko} \left(\frac{\varepsilon \operatorname{Lu}}{1 - \operatorname{Lu}} \frac{\sin \mu_n^* \frac{r}{R}}{\mu_n^* \frac{r}{R}} + \right.$$

$$\left. + \frac{\varphi_n^*}{\psi_n^*} \frac{\sin \sqrt{(\operatorname{Lu})}\, \mu_n^* \frac{r}{R}}{\mu_n^* \frac{r}{R}} \right) \exp \left(- \mu_n^{*2} \operatorname{Fo}^* \right),$$

$$\Theta = \frac{\theta_0 - \theta(r, \tau)}{\theta_0 - \theta_p} = 1 -$$

$$- \sum_{n=1}^{\infty} A_n^* \frac{\sin \mu_n^* \frac{r}{R}}{\mu_n^* \frac{r}{R}} \exp \left(- \mu_n^{*2} \operatorname{Fo}^* \right), \qquad (5.4)$$

where

$$\varphi_n = \operatorname{Ko} \operatorname{Lu} \frac{\operatorname{Bi}_m}{\operatorname{Bi}_q} \left\{ \left[1 - \varepsilon + \frac{\varepsilon}{\operatorname{Lu} - 1} (\operatorname{Bi}_q - 1) \right] \sin \frac{\mu_n}{\sqrt{\operatorname{Lu}}} - \right.$$

$$\left. - \left[1 - \varepsilon - \frac{\varepsilon}{\operatorname{Lu} - 1} \right] \frac{\mu_n}{\sqrt{\operatorname{Lu}}} \cos \frac{\mu_n}{\sqrt{\operatorname{Lu}}} \right\},$$

$$\varphi_n^* = \operatorname{Lu} \left\{ \left[1 - \varepsilon + \frac{\varepsilon}{\operatorname{Lu} - 1} (\operatorname{Bi}_q - 1) \right] \sin \mu_n^* - \right.$$

$$\left. - \left[1 - \varepsilon - \frac{\varepsilon}{\operatorname{Lu} - 1} \right] \mu_n^* \cos \mu_n^* \right\},$$

$$\psi_n = (\operatorname{Bi}_m - 1) \sin \frac{\mu_n}{\sqrt{\operatorname{Lu}}} + \frac{\mu_n}{\sqrt{\operatorname{Lu}}} \cos \frac{\mu_n}{\sqrt{\operatorname{Lu}}},$$

$$\psi_n^* = (\operatorname{Bi}_q - 1) \sin \sqrt{(\operatorname{Lu})}\, \mu_n^* + \sqrt{(\operatorname{Lu})}\, \mu_n^* \cos \sqrt{(\operatorname{Lu})}\, \mu_n^*,$$

μ_n and μ_n^* are determined from the characteristic equations

$$\tan \mu = - \frac{1}{\operatorname{Bi}_q - 1} \mu \text{ and } \tan \mu^* = - \frac{1}{\operatorname{Bi}_m - 1} \mu^*, \qquad (5.5)$$

and the coefficients A_n and A_n^* are determined by the equations

$$A_n = \frac{2(\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n - \sin \mu_n \cos \mu_n},$$

$$A_n^* = \frac{2(\sin \mu_n^* - \mu_n^* \cos \mu_n^*)}{\mu_n^* - \sin \mu_n^* \cos \mu_n^*}. \qquad (5.6)$$

TABLE 23. ROOTS OF THE CHARACTERISTIC EQUATIONS
$\cot \mu = 1/\mathrm{Bi}_q\, \mu$ and $\cot \mu^* = 1/\mathrm{Bi}_m\, \mu^*$

Bi_q / Bi_m	μ_1 / μ_1^*	μ_2 / μ_2^*	μ_3 / μ_3^*	μ_4 / μ_4^*	μ_5 / μ_5^*	μ_6 / μ_6^*
0	0·0000	3·1416	6·2832	9·4248	12·5664	15·7080
0·001	0·0316	3·1419	6·2833	9·4249	12·5665	15·7080
0·002	0·0447	3·1422	6·2835	9·4250	12·5665	15·7081
0·004	0·0632	3·1429	6·2838	9·4252	12·5667	15·7082
0·006	0·0774	3·1435	6·2841	9·4254	12·5668	15·7083
0·008	0·0893	3·1441	6·2845	9·4256	12·5670	15·7085
0·010	0·0998	3·1448	6·2848	9·4258	12·5672	15·7086
0·020	0·1410	3·1479	6·2864	9·4269	12·5680	15·7092
0·040	0·1987	3·1543	6·2895	9·4290	12·5696	15·7105
0·060	0·2425	3·1606	6·2927	9·4311	12·5711	15·7118
0·080	0·2791	3·1668	6·2959	9·4333	12·5727	15·7131
0·100	0·3111	3·1731	6·2991	9·4354	12·5743	15·7143
0·200	0·4328	3·2039	6·3148	9·4459	12·5823	15·7207
0·300	0·5218	3·2341	6·3305	9·4565	12·5902	15·7270
0·400	0·5932	3·2636	6·3461	9·4670	12·5981	15·7334
0·500	0·6533	3·2923	6·3616	9·4775	12·6060	15·7397
0·600	0·7061	3·3204	6·3770	9·4879	12·6139	15·7460
0·700	0·7506	3·3477	6·3923	9·4983	12·6218	15·7524
0·800	0·7910	3·3744	6·4074	9·5087	12·6296	15·7587
0·900	0·8274	3·4003	6·4224	9·5190	12·6375	15·7650
1·000	0·8603	3·4256	6·4373	9·5293	12·6453	15·7713
1·500	0·9882	3·5422	6·5097	9·5801	12·6841	15·8026
2·000	1·0769	3·6436	6·5783	9·6296	12·7223	15·8336
3·000	1·1925	3·8088	6·7040	9·7240	12·7966	15·8945
4·000	1·2646	3·9352	6·8140	9·8119	12·8678	15·9536
5·000	1·3138	4·0336	6·9096	9·8928	12·9352	16·0107
6·000	1·3496	4·1116	6·9924	9·9667	12·9988	16·0654
7·000	1·3766	4·1746	7·0640	10·0339	13·0584	16·1177
8·000	1·3978	4·2264	7·1263	10·0949	13·1141	16·1675
9·000	1·4149	4·4694	7·1806	10·1502	13·1660	16·2147
10·000	1·4289	4·3058	7·2281	10·2003	13·2142	16·2594
15·000	1·4729	4·4255	7·3959	10·3898	13·4078	16·4474
20·000	1·4961	4·4915	7·4954	10·5117	13·5420	16·5864
30·000	1·5202	4·5615	7·6057	10·6543	13·7085	16·7691
40·000	1·5325	4·5979	7·6647	10·7334	13·8048	16·8794
50·000	1·5400	4·6202	7·7012	10·7832	13·9666	16·9519
60·000	1·5451	4·6353	7·7259	10·8172	13·9094	17·0026
80·000	1·5514	4·6543	7·7573	10·8606	13·9644	17·0686
100·000	1·5552	4·6658	7·7764	10·8871	13·9981	17·1093
∞	1·5708	4·7124	7·8540	10·9956	14·1372	17·2788

TABLE 24. VALUES OF THE COEFFICIENTS A_n AND A_n^* FOR INFINITE PLATE

Bi_q / Bi_m	A_1 / A_1^*	A_2 / A_2^*	A_3 / A_3^*	A_4 / A_4^*	A_5 / A_5^*	A_6 / A_6^*
0·000	1·0000	—0·0000	0·0000	—0·0000	0·0000	—0·0000
0·001	1·0002	—0·0002	0·0000	—0·0000	0·0000	—0·0000
0·002	1·0004	—0·0004	0·0001	—0·0000	0·0000	—0·0000
0·004	1·0008	—0·0008	0·0002	—0·0001	0·0000	—0·0000
0·006	1·0012	—0·0012	0·0003	—0·0001	0·0001	—0·0000
0·008	1·0015	—0·0016	0·0004	—0·0002	0·0001	—0·0001
0·010	1·0020	—0·0020	0·0005	—0·0002	0·0001	—0·0001
0·020	1·0030	—0·0040	0·0010	—0·0004	0·0003	—0·0002
0·040	1·0065	—0·0080	0·0020	—0·0009	0·0005	—0·0003
0·060	1·0099	—0·0119	0·0030	—0·0013	0·0007	—0·0004
0·080	1·0130	—0·0158	0·0040	—0·0018	0·0010	—0·0006
0·100	1·0159	—0·0197	0·0050	—0·0022	0·0013	—0·0008
0·200	1·0312	—0·0381	0·0100	—0·0045	0·0025	—0·0016
0·300	1·0450	—0·0555	0·0148	—0·0067	0·0038	—0·0024
0·400	1·0581	—0·0719	0·0196	—0·0089	9·0050	—0·0032
0·500	1·0701	—0·0873	0·0243	—0·0110	0·0063	—0·0040
0·600	1·0813	—0·1025	0·0289	—0·0132	0·0075	—0·0048
0·700	1·0918	—0·1154	0·0335	—0·0153	0·0087	—0·0056
0·800	1·1016	—0·1282	0·0379	—0·0175	0·0100	—0·0064
0·900	1·1107	—0·1403	0·0423	—0·0196	0·0112	—0·0072
1·000	1·1192	—0·1517	0·0466	—0·0217	0·0124	—0·0080
1·500	1·1537	—0·2013	0·0667	—0·0318	0·0184	—0·0119
2·000	1·1784	—0·2367	0·0848	—0·0414	0·0241	—0·0157
3·000	1·2102	—0·2881	0·1154	—0·0589	0·0351	—0·0231
4·000	1·2287	—0·3215	0·1396	—0·0750	0·0451	—0·0300
5·000	1·2403	—0·3442	0·1588	—0·0876	0·0543	—0·0366
6·000	1·2478	—0·3604	0·1740	—0·0991	0·0626	—0·0427
7·000	1·2532	—0·3722	0·1861	—0·1089	0·0701	—0·0483
8·000	1·2569	—0·3812	0·1959	—0·1174	0·0768	—0·0535
9.000	1·2598	—0·3880	0·2039	—0·1246	0·0828	—0·0583
10·000	1·2612	—0·3934	0·2104	—0·1309	0·0881	—0·0676
15·000	1·2677	—0·4084	0·2320	—0·1514	0·1072	—0·0795
20·000	1·2699	—0·4147	0·2394	—0·1621	0·1182	—0·0901
30·000	1·2717	—0·4198	0·2472	—0·1718	0·1291	—0·1015
40·000	1·2723	—0·4127	0·2502	—0·1759	0·1340	—0·1069
50·000	1·2727	—0·4227	0·2517	—0·1779	0·1365	—0·1098
60.000	1·2728	—0·4232	0·2526	—0·1791	0·1379	—0·1115
80·000	1·2730	—0·4237	0·2535	—0·1803	0·1394	—0·1132
100·000	1·2731	—0·4239	0·2539	—0·1808	0·1405	—0·1141
∞	1·2732	—0·4244	0·2546	—0·1819	0·1415	—0·1157

The values of the characteristic roots are given in Table 25 and Fig. 19, and of the coefficients A_n and A_n^* in Table 26 and Fig. 20.

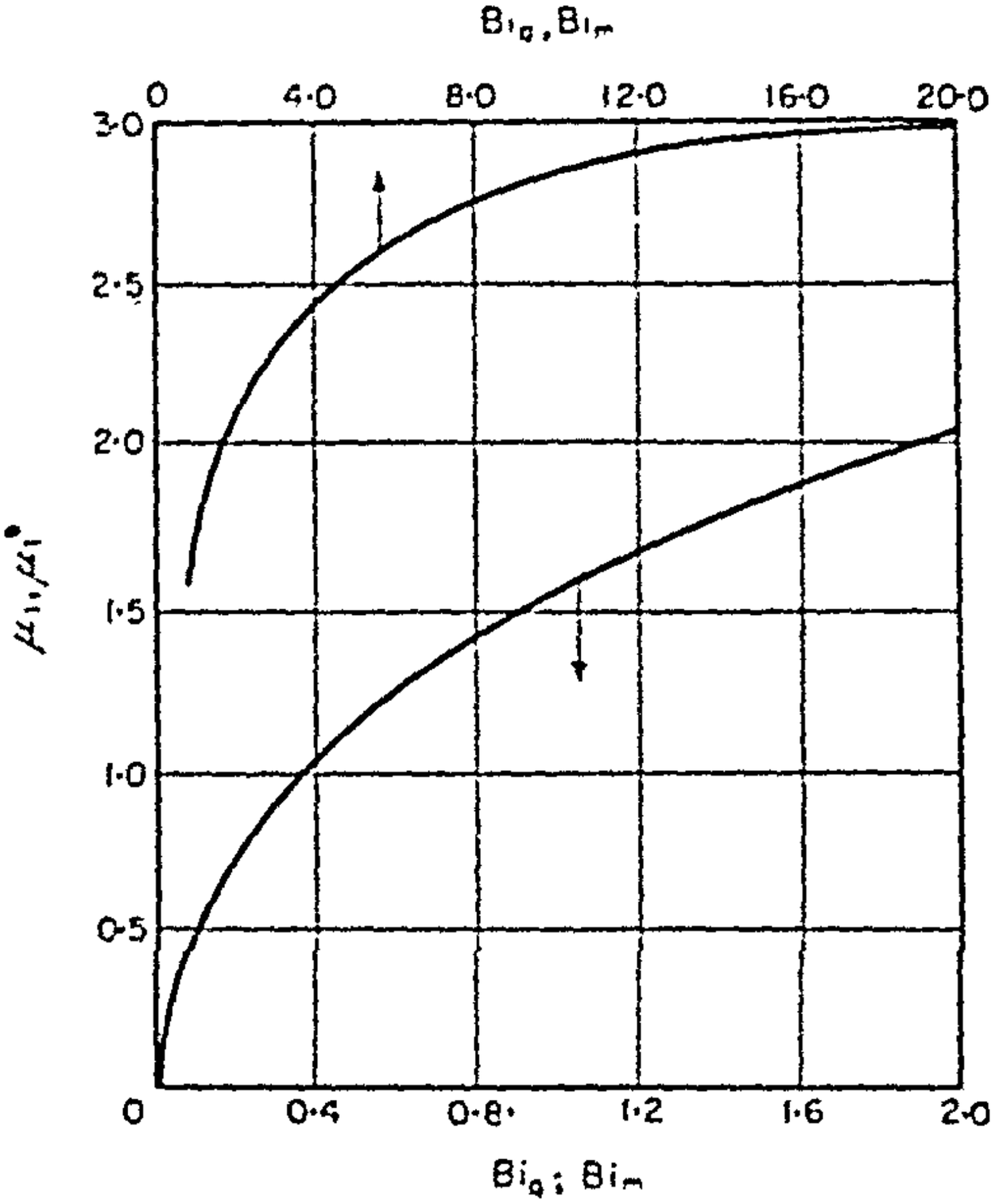


FIG. 19. Relation between the first roots of the characteristic equations $\tan \mu = -\frac{1}{\mathrm{Bi}_q - 1}\mu$; $\tan \mu^* = -\frac{1}{\mathrm{Bi}_m}\mu^*$ and the Biot criteria for a sphere

The general solution for heat and mass transfer in an infinite cylinder with constant initial conditions is:

$$T = \frac{t(r,\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} A_n \left(1 + \frac{\varphi_n}{\psi_n}\right) J_0\left(\mu_n \frac{r}{R}\right) \times$$

$$\times \exp\left(-\mu_n^2 \mathrm{Fo}\right) - \sum_{n=1}^{\infty} A_n^* \mathrm{Ko}\left[\frac{\varepsilon \mathrm{Lu}}{1 - \mathrm{Lu}} J_0\left(\mu_n^* \frac{r}{R}\right) + \right.$$

$$\left. + \frac{\varphi_n^*}{\psi_n^*} J_0\left(\mu_n^* \sqrt{(\mathrm{Lu})} \frac{r}{R}\right)\right] \exp\left(-\mu_n^{*2} \mathrm{Fo}^*\right),$$

TABLE 25. ROOTS OF THE CHARACTERISTIC EQUATIONS
$\tan \mu = -1/(\mathrm{Bi}_q - 1)\,\mu$ and $\tan \mu^* = -1/(\mathrm{Bi}_m - 1)\,\mu^*$

Bi_q / Bi_m	μ_1 / μ_1^*	μ_2 / μ_2^*	μ_3 / μ_3^*	μ_4 / μ_4^*	μ_5 / μ_5^*	μ_6 / μ_6^*
0·000	0·0000	4·4934	7·7253	10·9041	14·0662	17·2208
0·005	0·1224	4·4945	7·7259	10·9046	14·0666	17·2210
0·010	0·1730	4·4956	7·7265	10·9050	14·0669	17·2213
0·020	0·2445	4·4979	7·7278	10·9060	14·0676	17·2219
0·030	0·2991	4·5001	7·7291	10·9069	14·0683	17·2225
0·040	0·3450	4·5023	7·7304	10·9078	14·0690	17·2231
0·050	0·3854	4·5045	7·7317	10·9087	14·0697	17·2237
0·060	0·4217	4·5068	7·7330	10·9096	14·0705	17·2242
0·070	0·4551	4·5090	7·7343	10·9105	14·0712	17·2248
0·080	0·4860	4·5112	7·7356	10·9115	14·0719	17·2254
0·090	0·5150	4·5134	7·7369	10·9124	14·0726	17·2260
0·100	0·5423	4·5157	7·7382	10·9133	14·0733	17·2266
0·150	0·6609	4·5268	7·7447	10·9179	14·0769	17·2295
0·200	0·7593	4·5379	7·7511	10·9225	14·0804	17·2324
0·300	0·9208	4·5601	7·7641	10·9316	14·0875	17·2382
0·400	1·0528	4·5822	7·7770	10·9408	14·0946	17·2440
0·500	1·1656	4·6042	7·7899	10·9499	14·1017	17·2498
0·600	1·2644	4·6261	7·8028	10·9591	14·1088	17·2556
0·700	1·3525	4·6479	7·8156	10·9682	14·1159	17·2614
0·800	1·4320	4·6696	7·8284	10·9774	14·1230	17·2672
0·900	1·5044	4·6911	7·8412	10·9865	14·1301	17·2730
1·000	1·5708	4·7124	7·8540	10·9956	14·1372	17·2788
1·100	1·6320	4·7335	7·8667	11·0047	14·1443	17·2845
1·200	1·6887	4·7544	7·8794	11·0137	14·1513	17·2903
1·300	1·7414	4·7751	7·8920	11·0228	14·1584	17·2961
1·400	1·7906	4·7956	7·9046	11·0318	14·1654	17·3019
1·500	1·8366	4·8158	7·9171	11·0409	14·1724	17·3076
1·600	1·8798	4·8358	7·9295	11·0498	14·1795	17·3134
1·700	1·9203	4·8556	7·9419	11·0588	14·1865	17·3192
1·800	1·9586	4·8751	7·9542	11·0677	14·1935	17·3249
1·900	1·9947	4·8943	7·9665	11·0767	14·2005	17·3306
2·000	2·0288	4·9132	7·9787	11·0856	14·2075	17·3364
2·500	2·1746	5·0037	8·0385	11·1296	14·2421	11·3649
3·000	2·2889	5·0870	8·0962	11·1727	14·2764	17·3932
4·000	2·4557	5·2329	8·2045	11·2560	14·3434	17·4490
5·000	2·5704	5·3540	8·3029	11·3349	14·4080	17·5934
6·000	2·6537	5·4544	8·3914	11·4086	14·4699	17·5562
7·000	2·7165	5·5378	8·4703	11·4773	14·5258	17·6072
8·000	2·7654	5·6078	8·5406	11·5408	14·5847	17·6557
9·000	2·8044	5·6669	8·6031	11·5994	14·6374	17·7032
10·000	2·8363	5·7172	8·6587	11·6532	14·6870	17·7481
11·000	2·8628	5·7606	8·7083	11·7027	14·7335	17·7908
16·000	2·9476	5·9089	8·8898	11·8959	14·9251	17·9742
21·000	2·9930	5·9921	9·0019	12·0250	15·0625	18·1136
31·000	3·0406	6·0831	9·1294	12·1807	15·2389	18·3018

TABLE 25. *Contd.*

Bi_q / Bi_m	μ_1 / μ_1^*	μ_2 / μ_2^*	μ_3 / μ_3^*	μ_4 / μ_4^*	μ_5 / μ_5^*	μ_6 / μ_6^*
41·000	3·0651	6·1311	9·1987	12·2688	15·3417	18·4180
51·000	3·0801	6·1606	9·2420	12·3247	15·4090	18·4953
61·000	3·0901	6·1805	9·2715	12·3632	15·4559	18·5497
81·000	3·1028	6·2058	9·3089	12·4124	15·5164	18·6209
101·000	3·1105	6·2211	9·3317	10·4426	15·5537	18·6650
∞	3·1416	6·2832	9·4248	12·5664	15·7080	18·8496

TABLE 26. VALUES OF THE COEFFICIENTS A_n AND A_n^* FOR A SPHERE

Bi_q / Bi_m	A_1 / A_1^*	A_2 / A_2^*	A_3 / A_3^*	A_4 / A_4^*	A_5 / A_5^*	A_6 / A_6^*
0·000	1·0000	—0·0000	0·0000	—0·0000	0·0000	—0·0000
0·005	1·0025	—0·0023	0·0013	—0·0009	0·0007	—0·0006
0·010	1·0035	—0·0046	0·0026	—0·0018	0·0014	—0·0012
0·020	1·0055	—0·0091	0·0052	—0·0037	0·0029	—0·0023
0·030	1·0089	—0·0137	0·0078	—0·0055	0·0043	—0·0035
0·040	1·0121	—0·0182	0·0104	—0·0074	0·0057	—0·0047
0·050	1·0147	—0·0227	0·0130	—0·0092	0·0071	—0·0058
0·060	1·0181	—0·0273	0·0156	—0·0110	0·0085	—0·0070
0·070	1·0206	—0·0318	0·0183	—0·0129	0·0100	—0·0081
0·080	1·0239	—0·0363	0·0209	—0·0147	0·0114	—0·0093
0·090	1·0266	—0·0409	0·0235	—0·0166	0·0128	—0·0105
0·100	1·0297	—0·0454	0 0260	—0·0184	0·0142	—0·0116
0·150	1·0443	—0·0679	0·0390	—0·0276	0·0214	—0·0174
0·200	1·0592	—0·0894	0·0520	—0·0368	0·0285	—0·0232
0·300	1·0880	—0·1354	0·0779	—0·0551	0·0427	—0·0349
0·400	1·1164	—0·1781	0·1036	—0·0734	0·0569	—0·0465
0·500	1·1440	—0·2216	0·1292	—0·0916	0·0710	—0·0580
0·600	1·1713	—0·2633	0·1546	—0·1098	0·0852	—0·0696
0·700	1·1978	—0·3048	0·1799	—0·1270	0·0998	—0·0812
0·800	1·2237	—0·3455	0·2050	—0·1460	0·1134	—0·0927
0·900	1·2488	—0·3854	0·2299	—0·1640	0·1275	—0·1042
1·000	1·2732	—0·4244	0·2546	—0·1819	0·1415	—0·1157
1·100	1·2970	—0·4626	0·2792	—0·1997	0·1555	—0·1272
1·200	1·3200	—0·4999	0·3035	—0·2175	0·1694	—0·1387
1·300	1·3424	—0·5364	0·3276	—0·2352	0·1833	—0·1501
1·400	1·3640	—0·5720	0·3515	—0·2528	0·1972	—0·1616
1·500	1·3848	—0·6067	0·3752	—0·2703	0·2110	—0·1730
1·600	1·4051	—0·6405	0·3986	—0·2878	0·2248	—0·1843
1·700	1·4247	—0·6735	0·4218	—0·3051	0·2385	—0·1957
1·800	1·4436	—0·7063	0·4447	—0·3228	0·2522	—0·2078
1·900	1·4618	—0·7368	0·4674	—0·3395	0·2659	—0·2183
2·000	1·4793	—0·7673	0·4898	—0·3565	0·2795	—0·2296

TABLE 26. *Contd.*

Bi_q / Bi_m	A_1 / A_1^*	A_2 / A_2^*	A_3 / A_3^*	A_4 / A_4^*	A_5 / A_5^*	A_6 / A_6^*
2·500	1·5579	—0·9073	0·5980	—0·4365	0·3449	—0·2855
3·000	1·6223	—1·0288	0·6993	—0·5204	0·4122	—0·3405
4·000	1·7201	—1·2253	0·8811	—0·6719	0·5384	—0·4476
5·000	1·7870	—1·3733	1·0363	—0·8095	0·6570	—0·5501
6·000	1·8338	—1·4860	1·1673	—0·9333	0·7702	—0·6428
7·000	1·8673	—1·5731	1·2776	—1·0437	0·8695	—0·7398
8·000	1·8920	—1·6409	1·3703	—1·1415	0·9633	—0·8264
9·000	1·9106	—1·6949	1·4482	—1·2280	1·0489	—0·9073
10·000	1·9249	—1·7381	1·5141	—1·3042	1·1269	—0·9027
11·000	1·9364	—1·7732	1·5698	—1·3713	1·1977	—1·0527
16·000	1·9663	—1·8766	1·7489	—1·6058	1·4633	—1·3305
21·000	1·9801	—1·9235	1·8385	—1·7360	1·6256	—1·5149
31·000	1·9905	—1·9626	1·9186	—1·8616	1·7950	—1·7225
41·000	1·9948	—1·9780	1·9515	—1·9161	1·8732	—1·8263
51·000	1·9964	—1·9856	1·9860	—1·9441	1·9145	—1·8802
61·000	1·9974	—1·9901	1·9773	—1·9601	1·9387	—1·9135
81·000	1·9985	—1·9942	1·9869	—1·9769	1·9644	—1·9482
101·000	1·9993	—1·9962	1·9915	—1·9850	1·9767	—1·9667
∞	2·000	—2·0000	2·0000	—2·0000	2·0000	—2·0000

μ_n and μ_n^* are the roots of the characteristic equations

$$\frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{Bi_q} \text{ and } \frac{J_0(\mu^*)}{J_1(\mu^*)} = \frac{\mu^*}{Bi_m}, \tag{5.8}$$

and the coefficients A_n and A_n^* are determined by the equations

$$A_n = \frac{2J_1(\mu_n)}{\mu_n[J_0^2(\mu_n) + J_1^2(\mu_n)]}, A_n^* = \frac{2J_1(\mu_n^*)}{\mu_n^*[J_0^2(\mu_n^*) + J_1^2(\mu_n^*)]}. \tag{5.9}$$

Here J_0 and J_1 are Bessel functions of the first kind, of zero and first orders.

The roots of the characteristic equations (5.8) are given in Table 27 and the values of the coefficients A_n and A_n^* in Table 28. A graphical representation of the same magnitudes is given in Figs. 21 and 22 respectively.

(b) *Absence of thermal transfer of mass and phase change* ($\delta = 0$, $\varepsilon = 0$).

For a plate

$$T = \frac{t(x, \tau) - t_0}{t_c - t_0} = 1 -$$

$$- \sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{Ko}\sqrt{\mathrm{Lu}}\,\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\,\mu_n \sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}}}{\mathrm{Bi}_m \cos\frac{\mu_n}{\sqrt{\mathrm{Lu}}} - \frac{\mu_n}{\sqrt{\mathrm{Lu}}}\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}}} \right] \times$$

$$\times \cos\mu_n \frac{x}{R} \exp\left(-\mu_n^2\,\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} A_n^* \frac{\mathrm{Ko}\,\mathrm{Lu}\,\mu_n^* \sin\mu_n^* \cos\left(\sqrt{(\mathrm{Lu})}\,\mu_n^* \frac{x}{R}\right)}{\mathrm{Bi}_q \cos\sqrt{(\mathrm{Lu})}\,\mu_n^* - \sqrt{(\mathrm{Lu})}\,\mu_n^* \sin\sqrt{(\mathrm{Lu})}\,\mu_n^*} \exp\left(\mu_n^{*2}\,\mathrm{Fo}^*\right).$$

For a sphere

$$T = \frac{t(r,\tau) - t_0}{t_c - t_0} = 1 -$$

$$- \sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\left(\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}} - \frac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)}{(\mathrm{Bi}_m - 1)\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}} + \frac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}}}} \right] \times$$

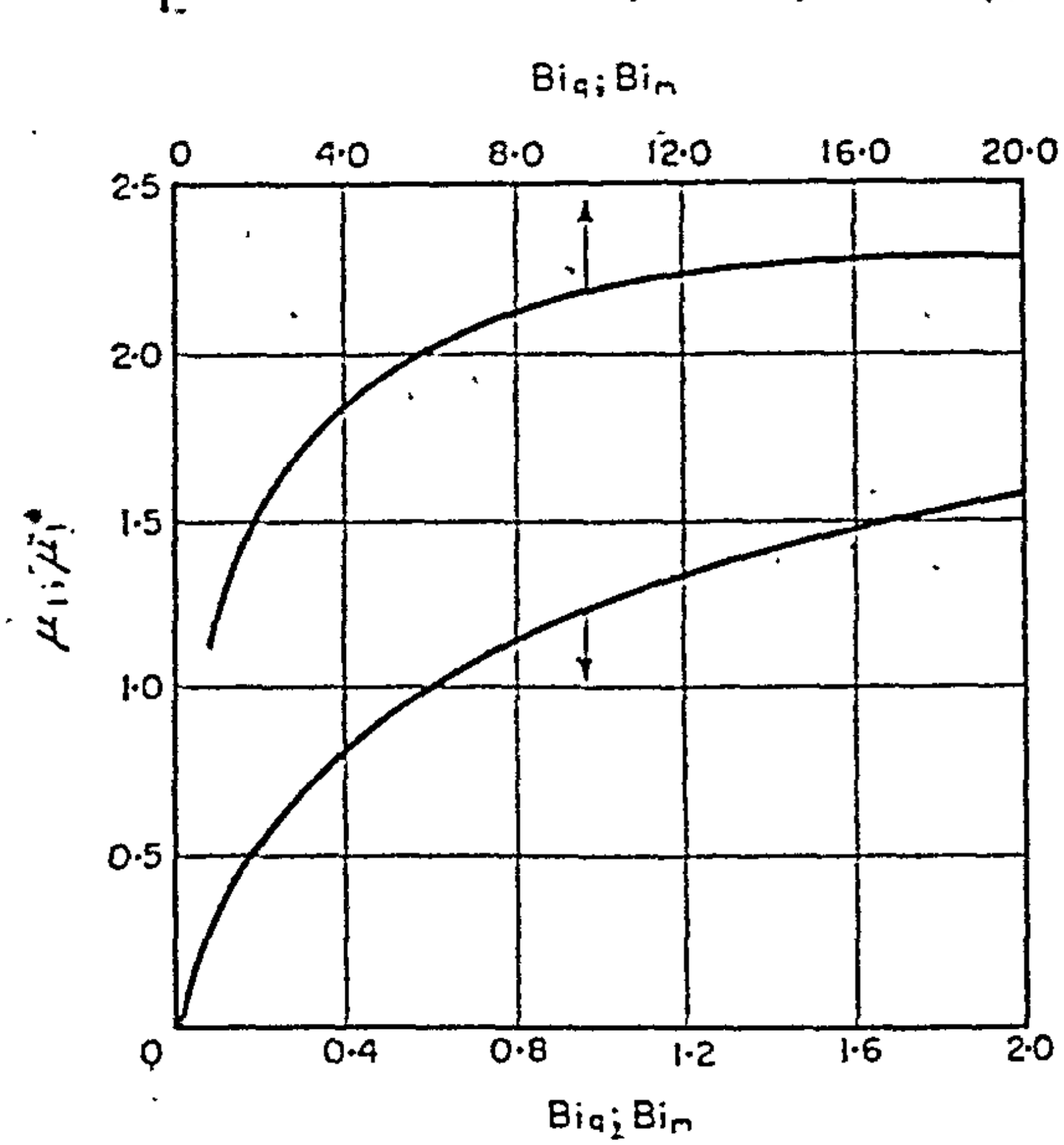


FIG. 21. Relation between the first roots of the characteristic equations $\frac{J_0(\mu)}{J^1(\mu)} = \frac{\mu}{\mathrm{Bi}_q}$ and $\frac{J_0(\mu^*)}{J_1(\mu^*)} = \frac{\mu^*}{\mathrm{Bi}_m}$ and the Biot criteria for an infinite cylinder

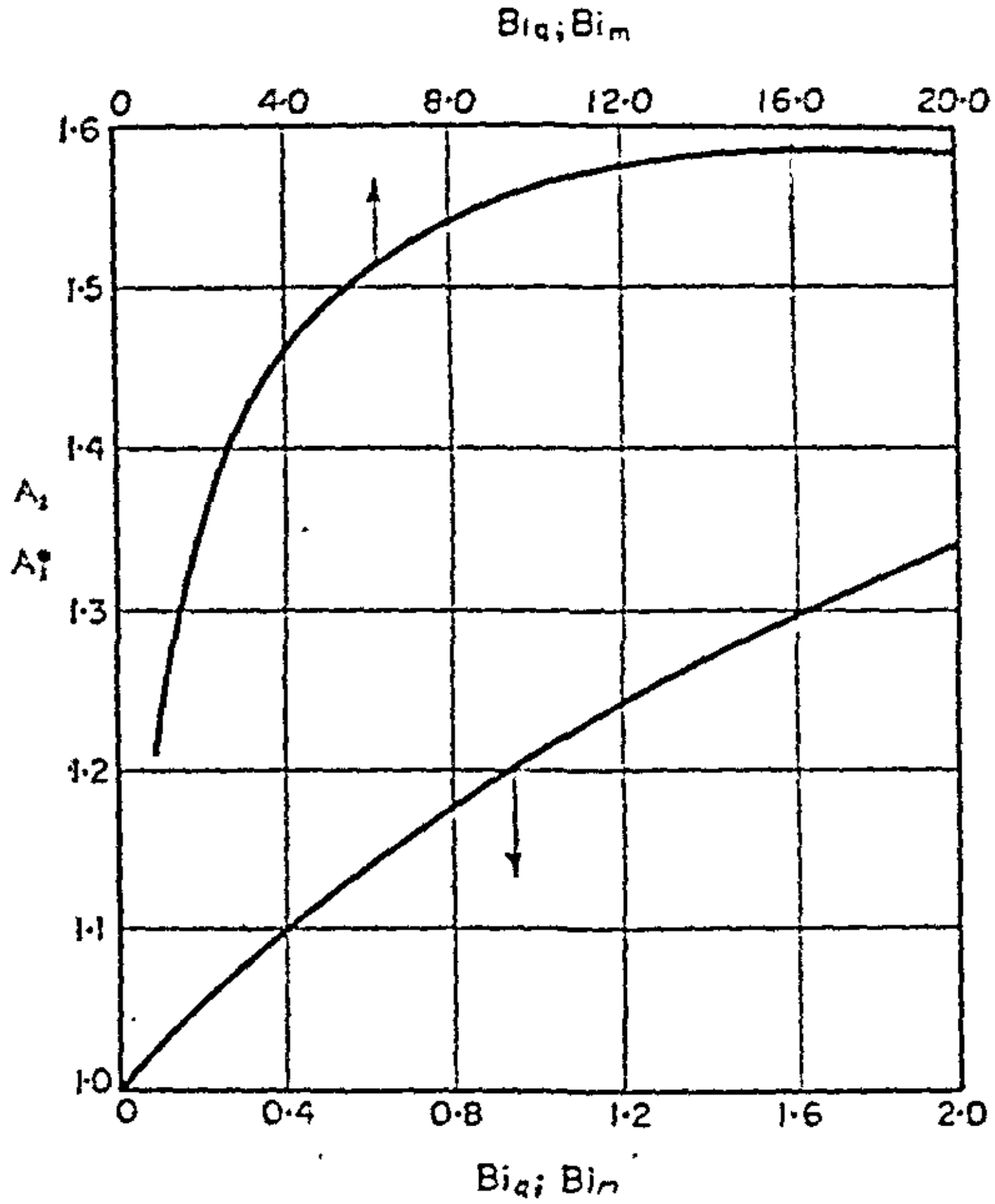


FIG. 22. Relation between A_1, A_1^* and the Biot criteria for an infinite cylinder

$$\times \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left(-\mu_n^2 \mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} A_n^* \frac{\mathrm{Ko}\,\sqrt{\mathrm{Lu}}\,\mathrm{Bi}_m \sin \mu_n^*}{(\mathrm{Bi}_q - 1)\sin \sqrt{(\mathrm{Lu})}\,\mu_n^* + \sqrt{(\mathrm{Lu})}\,\mu_n^* \cos \sqrt{(\mathrm{Lu})}\,\mu_n^*} \times$$

$$\times \frac{\sin \sqrt{(\mathrm{Lu})}\,\mu_n^* \frac{r}{R}}{\mu_n^* \frac{r}{R}} \exp\left(-\mu_x^{*2}\,\mathrm{Fo}^*\right).$$

For a cylinder

$$T = \frac{t(r, \tau) - t_0}{t_c - t_0}$$

$$= 1 - \sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{Ko}\sqrt{\mathrm{Lu}}\,\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\,\mu_n J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)}{\mathrm{Bi}_m J_0\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right) - \frac{\mu_n}{\sqrt{\mathrm{Lu}}} J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)} \right] \times$$

$$\times J_0\left(\mu_n \frac{r}{R}\right) \exp\left(-\mu_n^2 \mathrm{Fo}\right) -$$

$$-\sum_{n=1}^{\infty} A_n^* \frac{\mathrm{Ko}\,\mathrm{Lu}\,\mu_n^* J_1(\mu_n^*)\, J_0\left(\mu_n^* \sqrt{(\mathrm{Lu})}\, \frac{r}{R}\right)}{\mathrm{Bi}_q J_0\left[\sqrt{(\mathrm{Lu})}\,\mu_n^*\right] - \sqrt{(\mathrm{Lu})}\,\mu_n^* J_1\left[\sqrt{(\mathrm{Lu})}\,\mu_n^*\right]} \exp\left(-\mu_n^{*2}\mathrm{Fo}^*\right).$$

TABLE 27. ROOTS OF THE CHARACTERISTIC EQUATIONS

$J_0(\mu)/J_1(\mu) = \mu/\mathrm{Bi}_q$ and $J_0(\mu^*)/J_1(\mu^*) = \mu^*/\mathrm{Bi}_m$

Bi_q / Bi_m	μ_1 / μ_1^*	μ_2 / μ_2^*	μ_3 / μ_3^*	μ_4 / μ_4^*	μ_5 / μ_5^*	μ_6 / μ_6^*
0·00	0·0000	3·8317	7·0156	10·1735	13·3237	16·4706
0·01	0·1412	3·8343	7·0170	10·1754	13·3244	16·4712
0·02	0·1995	3·8369	7·0184	10·1754	13·3252	16·4718
0·04	0·2814	3·8421	7·0213	10·1774	13·3267	16·4731
0·06	0·3438	3·8473	7·0241	10·1794	13·3282	16·4743
0·08	0·3960	3·8525	7·0270	10·1813	13·3297	16·4755
0·10	0·4417	3·8577	7·0298	10·1833	13·3312	16·4767
0·15	0·5376	3·8706	7·0369	10·1882	13·3349	16·4797
0·20	0·6170	3·8835	7·0440	10·1931	13·3387	16·4828
0·30	0·7465	3·9091	7·0582	10·2029	18·3462	16·4888
0·40	0·8516	3·9344	7·0723	10·2127	13·3537	16·4949
0·50	0·9408	3·9594	7·0864	10·2225	13·3611	16·5010
0·60	1·0184	3·9841	7·1004	10·2322	13·3686	16·5070
0·70	1·0873	4·0085	7·1143	10·2419	13·3761	16·5131
0·80	1·1490	4·0325	7·1282	10·2519	13·3835	16·5191
0·90	1·2048	4·0562	7·1421	10·2613	13·3910	16·5251
1·00	1·2558	4·0795	7·1558	10·2710	13·3984	16·5312
1·50	1·4569	4·1902	7·2233	10·3188	13·4353	16·5612
2·00	1·5994	4·2910	7·2994	10·3658	13·4719	16·5910
3·00	1·7887	4·4634	7·4103	10·4566	13·5434	16·6499
4·00	1·9081	4·6018	7·5201	10·5423	13·6125	16·7073
5·00	1·9898	4·7131	7·6177	10·6223	13·6786	16·7630
6·00	2·0490	4·8033	7·7039	10·6964	13·7414	16·8168
7·00	2·0937	4·8772	7·7797	10·7646	13·8008	16·8684
8·00	2·1286	4·9384	7·8464	10·8271	13·8566	16·9179
9·00	2·1566	4·9897	7·9051	10·8842	13·9090	16·9650
10·00	2·1795	5·0332	7·9569	10·9363	13·9580	17·0099
15·00	2·2509	4·1773	8·1422	11·1367	14·1576	17·2008
20·00	2·2880	5·2568	8·2534	11·2677	14·2983	17·3442
30·00	2·3261	5·3410	8·3771	11·4221	14·4748	17·5348
40·00	2·3455	5·3846	8·4432	11·5081	14·5774	17·6508
50·00	2·3572	5·4112	8·4840	11·5621	14·6433	17·7272
60·00	2·3651	5·4291	8·5116	11·5990	14·6889	17·7807
80·00	2·3750	5·4516	8·5466	11·6461	14·7475	17·8502
100·00	2·3809	5·4652	8·5678	11·6747	14·7834	17·8931
∞	2·4048	5·5201	8·6537	11·7915	14·9309	18·0711

TABLE 28. VALUES OF THE COEFFICIENTS A_n AND A_n^* FOR AN INFINITE CYLINDER

Bi_q / Bi_m	$+A_1$ / $+A_1^*$	$-A_2$ / $-A_2^*$	$+A_3$ / $+A_3^*$	$-A_4$ / $-A_4^*$	$+A_5$ / $+A_5^*$	$-A_6$ / $-A_6^*$
0·00	1·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·01	1·0031	0·0034	0·0013	0·0008	0·0005	0·0004
0·02	1·0049	0·0067	0·0027	0·0015	0·0010	0·0007
0·04	1·0102	0·0135	0·0052	0·0031	0·0021	0·0015
0·06	1·0150	0·0201	0·0081	0·0046	0·0031	0·0023
0·08	1·0199	0·0268	0·0110	0·0062	0·0041	0·0030
0·10	1·0245	0·0333	0·0135	0·0077	0·0051	0·0037
0·15	1·0366	0·0497	0·0202	0·0116	0·0077	0·0056
0·20	1·0482	0·0658	0·0269	0·0154	0·0103	0·0075
0·30	1·0711	0·0972	0·0401	0·0231	0·0155	0·0112
0·40	1·0931	0·1277	0·0582	0·0307	0·0205	0·0150
0·50	1·1142	0·1571	0·0662	0·0383	0·0256	0·0187
0·60	1·1345	0·1857	0·0790	0·0458	0·0307	0·0224
0·70	1·1539	0·2132	0·0917	0·0533	0·0358	0·0261
0·80	1·1724	0·2398	0·1043	0·0608	0·0408	0·0298
0·90	1·1902	0·2654	0·1167	0·0682	0·0459	0·0335
1·00	1·2071	0·2901	0·1289	0·0756	0·0509	0·0372
1·50	1·2807	0·4008	0·1877	0·1117	0·0756	0·0554
2·00	1·3377	0·4923	0·2422	0·1404	0·0998	0·0732
3·00	1·4192	0·6309	0·3384	0·2114	0·1463	0·1084
4·00	1·4698	0·7278	0·4184	0·2699	0·1898	0·1420
5·00	1·5029	0·7973	0·4842	0·3220	0·2301	0·1735
6·00	1·5253	0·8484	0·5382	0·3679	0·2672	0·2038
7·50	1·5409	0·8869	0·5825	0·4089	0·3010	0·2317
8·00	1·5523	0·9225	0·6189	0·4430	0·3316	0·2579
9·00	1·5611	0·9393	0·6491	0·4735	0·3593	0·2826
10·00	1·5677	0·9575	0·6784	0·5000	0·3843	0·3042
15·00	1·5853	1·0091	0·7519	0·5901	0·4760	0·3913
20·00	1·5918	1·0309	0·7889	0·6382	0·5303	0·4461
30·00	1·5964	1·0488	0·8195	0·6827	0·5853	0·5062
40·00	1·5988	1·0550	0·8335	0·7018	0·6133	0·5390
50·00	1·5995	1·0587	0·8396	0·7112	0·6227	0·5544
60·00	1·6009	1·0589	0·8428	0·7165	0·6301	0·5642
80·00	1·6012	1·0599	0·8463	0·7212	0·6398	0·5770
100·00	1·6014	1·0631	0·8505	0·7245	0·6415	0·5850
∞	1·6021	1·0648	0·8558	0·7296	0·6485	0·5896

The expressions for fields of mass content are determined by equations (5.1), (5.4) and (5.7); the roots of the characteristic equations by equations (5.2), (5.5) and (5.8); the coefficients of heat and mass transfer by equations (5.3), (5.6) and (5.9).

In the absence of mass exchange (Ko = 0) the solutions written above are transformed into the ordinary classical solutions for the distribution of temperature in the plate, sphere and cylinder.

(c) *The rate of removal of mass from the body is constant; no thermal transfer of mass.* ($\delta = 0$, $q_m = \beta[\theta(R, \tau) - \theta_p] = \text{const.}$). This problem can serve as an analogy to the mathematical description of the period of constant rate of drying. Its solution, given by F. M. Polonskaya, was the first step in the working out of the system of differential equations of heat and mass transfer (I), (II).

The solution of the system of equations for the distribution of potential of mass transfer in an infinite plate has the form:

$$\frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[\mathrm{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2 \pi^2} \cos n\pi \frac{x}{R} \exp\left(-n^2 \pi^2 \mathrm{Fo}^*\right) \right], \quad (5.10)$$

where $\mathrm{Ki}_m = q_m R / \lambda_m (\theta_0 - \theta_p)$ is the mass exchange criterion of Kirpichev.

With increase in the non-dimensional time (Fo*), the terms of the infinite series (5.10) rapidly diminish, so that beginning with $\mathrm{Fo}^* = \mathrm{FoLu} = a_m \tau / R^2 \geqslant 0{\cdot}54$ they may be neglected in comparison with the first two members of (5.10), i.e. a quasi-stationary condition is set up in the material:

$$\frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[\mathrm{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) \right] \text{ where } \mathrm{Fo}^* \geqslant 0{\cdot}54 .$$

In this case the solution for the temperature field will be:

$$\frac{t(x, \tau) - t_0}{t_m - t_0} = 1 - \frac{1}{2} \varepsilon \,\mathrm{Lu}\,\mathrm{Ko}\,\mathrm{Ki}_m \left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) -$$

$$- \sum_{n=1}^{\infty} A_n \left(1 - \frac{1}{\mu_n^2} \varepsilon \, \mathrm{Lu} \, \mathrm{Ko} \, \mathrm{Ki}_m\right) \cos \mu_n \frac{x}{R} \exp\left(- \mu_n^2 \, \mathrm{Fo}\right). \quad (5.11)$$

Here μ_n and A_n are determined by expressions (5.1) and (5.3) and $t_m = t_c - (1 - \varepsilon)\varrho q_m/\alpha$. Beginning from the value of the criterion $\mathrm{Fo} \geqslant 0{\cdot}32$ the solution (5.11) becomes simpler and the expression for the non-dimensional temperature takes the form

$$\frac{t(x, \tau) - t_0}{t_m - t_0} = 1 - \frac{1}{2} \varepsilon \, \mathrm{Lu} \, \mathrm{Ko} \, \mathrm{Ki}_m \left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right).$$

Hence the temperature field becomes stationary. F. M. Polonskaya's experiments confirmed this temperature distribution in moist material.

Closer examination of the last relation shows that the criterion of phase change is

$$\varepsilon = \frac{2\lambda}{\varrho q_m R}(t_n - t_u).$$

This means that for a known temperature drop and with known thermo-physical properties of the material we can determine ε, and hence the flux of matter in the vapour state. According to the experiments of F. M. Polonskaya and P. D. Lebedev, for a gypsum plate $\varepsilon = 0{\cdot}045$; for clay, 0·75 to 1; for sand, 0·2 to 0·4; and for wood, 0·09 to 0·2.

(d) *Generalized boundary conditions of the third kind; no thermal transfer of mass.* In the foregoing problems we considered the analytical solutions of different cases of heat and mass transfer. In these cases the rate of removal of matter from the material was either constant, $q_m = \text{const.}$ or dependent on the transfer potential of mass at the surface of the body, $q_m = \beta\,[\theta(R, \tau) - \theta_p]$. The term in the boundary conditions which corresponds to the rate of removal of mass into the surrounding medium may have this or another form, depending on the conditions of the process; q_m may depend in various ways on the coordinates and the time. Hence it is extremely desirable to have a single solution valid for the

whole process of heat and mass transfer. In the case of $q_m = q_m(\tau)$, $\delta = 0$ and regular conditions, such a solution for an infinite plate was obtained by A. V. Luikov and M. S. Smirnov. The boundary conditions for this problem are:

$$-\lambda_q \frac{\partial t(R,\tau)}{\partial x} + \alpha [t_c - t(R,\tau)] - (1-\varepsilon)\varrho\, q_m(\tau) = 0\,, \quad (5.12)$$

$$\lambda_m \frac{\partial \theta(R,\tau)}{\partial x} + q_m(\tau) = 0\,, \quad (5.13)$$

$$t(x,0) = t_0 = \text{const.} \quad \theta(x,0) = \theta_0 = 0\,, \quad (5.14)$$

$$\frac{\partial \theta(0,\tau)}{\partial x} = \frac{\partial t(0,\tau)}{\partial x} = 0\,. \quad (5.15)$$

The solution of the system of equations (I), (II) obtained by the method of integral transformation under the boundary conditions (5.12) to (5.15) for the field of transfer potential of mass has the form

$$\frac{\theta(x,\tau)}{\theta_0} = 1 - \int_0^{\tau} \frac{a_m}{R^2} \mathrm{Ki}_m(\eta)\, d\eta + \mathrm{Ki}_m(\tau)\, \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) +$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2\pi^2} \cos n\pi \frac{x}{R} \exp\left(-n^2\pi^2\, \mathrm{Fo}^*\right) \times$$

$$\times \left\{\mathrm{Ki}_m(0) + \int_0^{\tau} \exp\left(n^2\pi^2 \frac{a_m \eta}{R^2}\right) \frac{\mathrm{Ki}_m(\eta)}{d\eta}\, d\eta\right\}. \quad (5.16)$$

After a certain interval of time, i.e. for $\mathrm{Fo}^* > \mathrm{Fo}_1^*$, we can neglect the infinite series in formula (5.16). Then the expression will contain only the first two terms.

The temperature field of an infinite plate for steady conditions may be written:

$$\frac{t(x,\tau) - t_0}{t_c - t_0}$$

$$= 1 - \mathrm{Ki}_m(\tau)\, \mathrm{Ko}\, \mathrm{Lu}\, \frac{t}{t_c - t_0}\left[0{\cdot}5\,\varepsilon\left(1 - \frac{x^2}{R^2}\right) + \frac{1}{\mathrm{Bi}_q}\right]. \quad (5.17)$$

In the formulae (5.16) and (5.17),

$\mathrm{Ki}_m(\tau) = q_m(\tau) R/\lambda_m\, \theta_0$ is the Kirpichev mass exchange criterion,

$\mathrm{Ko} = \varrho c_m \vartheta_0 / c_a t_a$ is Kossovich's criterion.

Let us find the mean integral values for the transfer potentials. We have

$$\frac{\bar{\vartheta}(\tau)}{\vartheta_0} = 1 - \int_0^\tau \frac{a_m}{R^2} \mathrm{Ki}_m(\eta)\, d\eta\,,$$

$$\frac{\bar{t}(\tau) - t_0}{t_c - t_0} = 1 - \mathrm{Ki}_m(\tau)\, \mathrm{Ko}\, \mathrm{Lu}\, \frac{t_c}{t_c - t_0} \left[\frac{1}{3}\varepsilon + \frac{1}{\mathrm{Bi}_q}\right].$$

We obtain from these relations, for the rate of change of the transfer potential of mass (for example, in the drying process, for the rate of drying), the following expression:

$$-\frac{d\bar{u}(\tau)}{d\tau} = \frac{u_0 a_m [t_c - \bar{t}(\tau)]}{R^2\, \mathrm{Ko}\, \mathrm{Lu}\, t_c \left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)}$$

$$= \frac{\lambda_q [\bar{t}_c - \bar{t}(\tau)]}{\varrho\gamma_0 R^2 \left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)}. \qquad (5.18)$$

The relation (5.18) is of great practical interest, since it enables us to determine the rate of drying from the mean temperature of the body and the coefficients of heat transfer, in other words to reduce the calculation of the rate of mass transfer to the calculation of heat transfer. From the relations given above we can also obtain a formula for the mass exchange criterion of Kirpichev:

$$\mathrm{Ki}_m(\tau) = \frac{\lambda_q [t_c - \bar{t}(\tau)]}{\varrho a_m \gamma_0 u_0 \left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)},$$

from which it follows that the rate of evaporation depends on the criterion of phase change, and the transfer coefficients of heat and mass.

§ 6. Influence of the Criteria of Heat and Mass Exchange on the Process of Heat and Mass Transfer

The solutions of the system of differential equations of heat and mass transfer depend on a large group of heat and mass exchange criteria of similarity. These criteria do not, however,

all influence the course of the process to the same extent. Analysis of the results of a large number of calculations of equations (I – VI) indicated that certain criteria (Bi_m and Pn) primarily influence the mass exchange characteristics of the heat and mass transfer, whereas others (Bi_q, Ko, ε, and W) influence mainly the heat exchange.

The most considerable influence on the processes of heat and mass transfer is exercised by the criterion Lu, which may also be called the criterion of the inter-relation of heat and mass transfer. Fundamentally, Lu characterizes the inertial properties of the transfer potential field of mass in comparison with the inertial properties of the temperature field. It will be seen from Fig. 23 that at small values of Lu the distribution of heat potential in the material leads the distribution of the mass transfer potential. For Lu $\approx$ 1·0 they

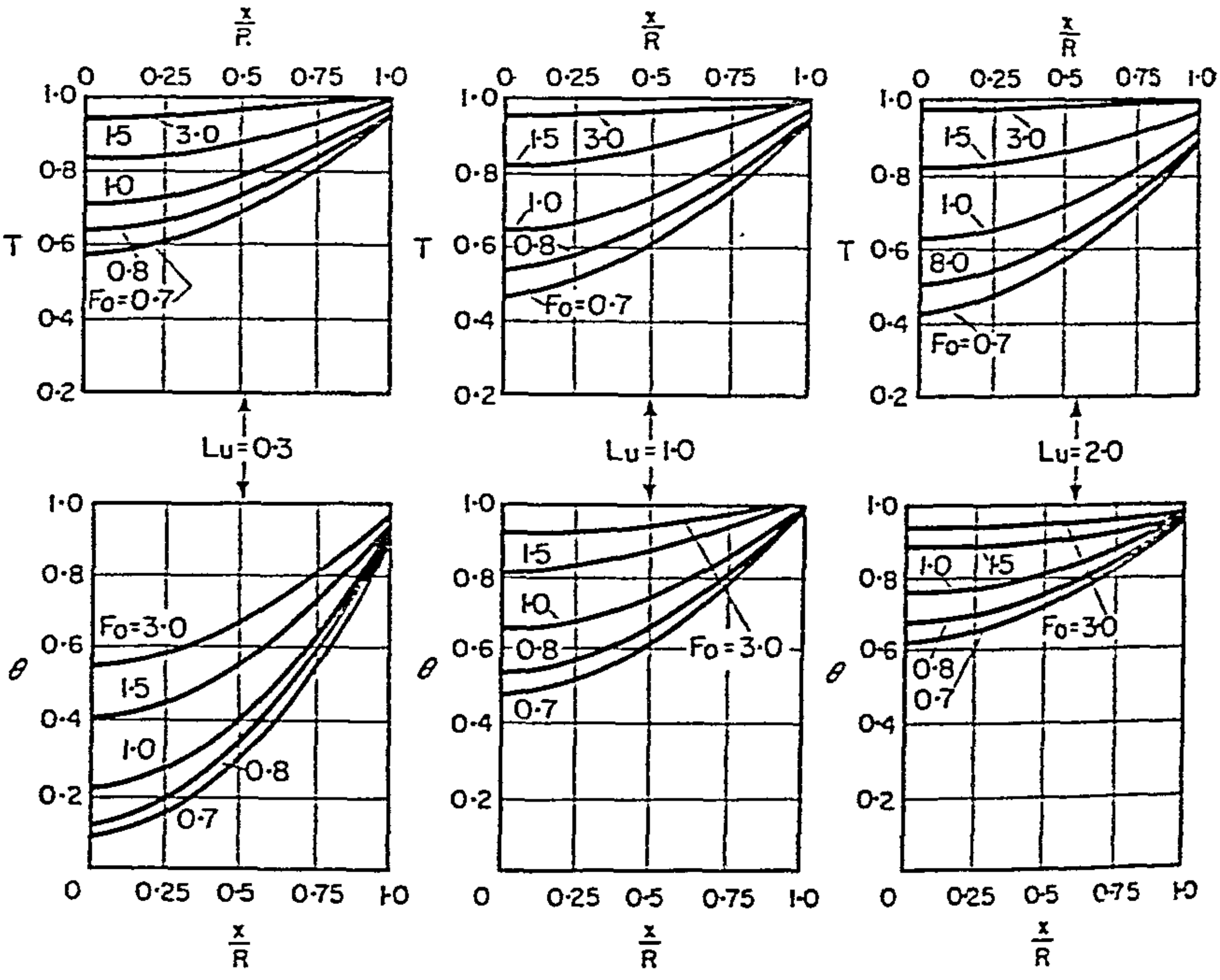


FIG. 23. *Distribution of transfer potentials through the material in relation to Lu*

are approximately comparable, and for Lu $>$ 1·0 the distribution of mass potential begins to precede that of temperature. However, no complete symmetry is observed in the transfer characteristics in relation to Lu = 1·0. For comparison let us consider two cases: Lu = 0·5 and Lu = 2·0, i.e. when in one case the coefficient of temperature conductivity is twice the coefficient of potential conductivity, of mass transfer, and in another when the former is half the latter.

TABLE 29.

Fo	$\bar{T} : \bar{\Theta}$	$\bar{\Theta} : \bar{T}$	$\frac{d\bar{T}}{d\mathrm{Fo}} : \frac{d\bar{\Theta}}{d\mathrm{Fo}}$	$\frac{d\bar{\Theta}}{d\mathrm{Fo}} : \frac{d\bar{T}}{d\mathrm{Fo}}$	$\bar{T} : \bar{\Theta}$	$\frac{d\bar{T}}{d\mathrm{Fo}} : \frac{d\bar{\Theta}}{d\mathrm{Fo}}$
	Lu = 0·5	Lu = 2·0	Lu = 0·5	Lu = 2·0	Lu = 1·0	
0·4	1·44	1·61			0·906	
0·6	1·41	1·35	1·23	0·63	0·946	1·10
0·8	1·35	1·22	0·22	0·60	0·965	1·03
1·0	1·29	1·15	0·79	0·59	0·976	1·03

It is clear from Table 29 and Fig. 24 that the difference for the mean non-dimensional potentials amounts to $\pm$ 12

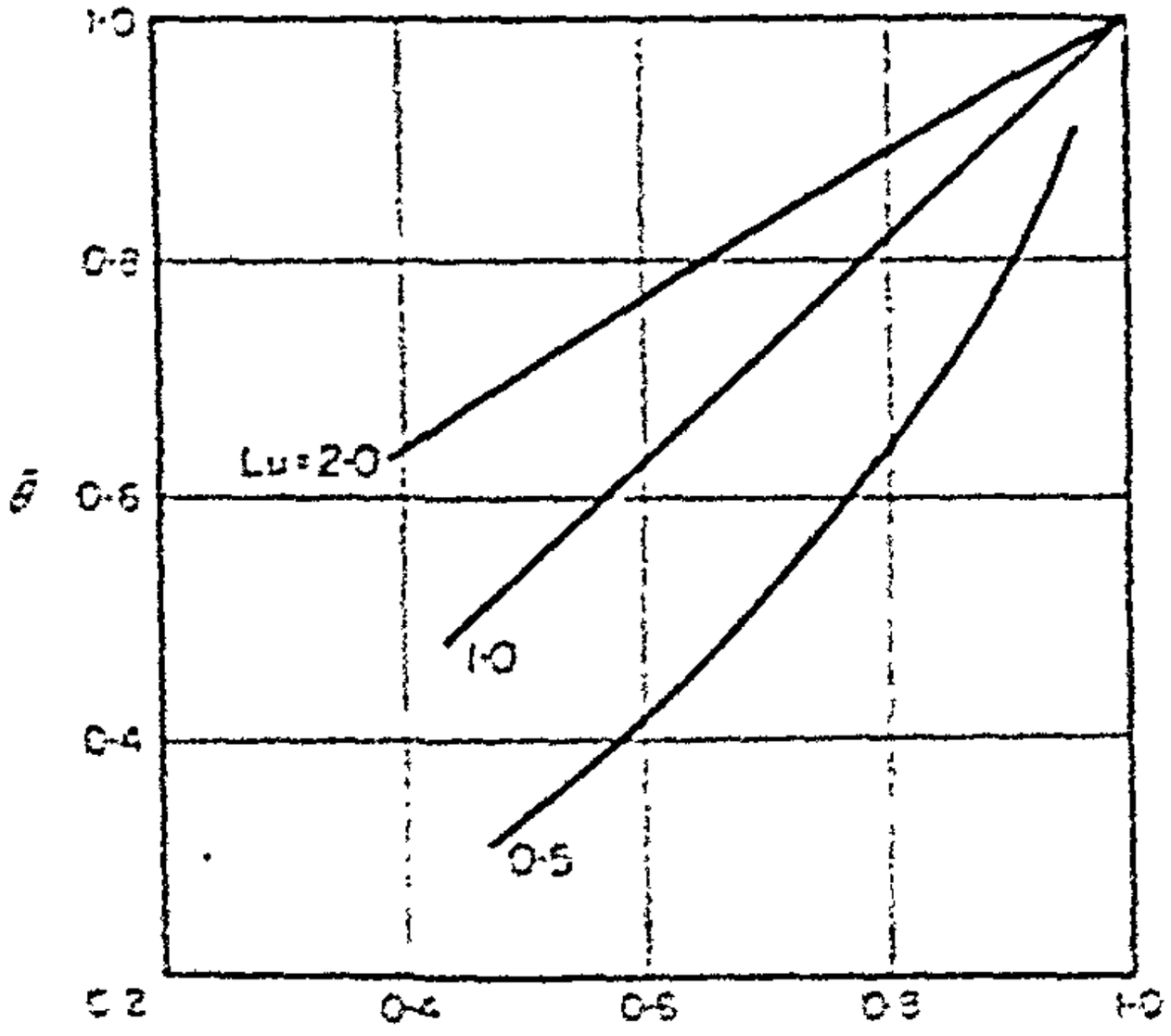


FIG. 24. Symmetry of $\bar{T}$ and $\bar{\Theta}$ in relation to Lu = 1·0

per cent, and for the ratio of their rates of change to 200 per cent. No complete equality of the transfer of heat and mass is observed even for Lu = 1·0, but it is begins at Lu < 1·0. With increasing Lu for the same value of the non-dimensional

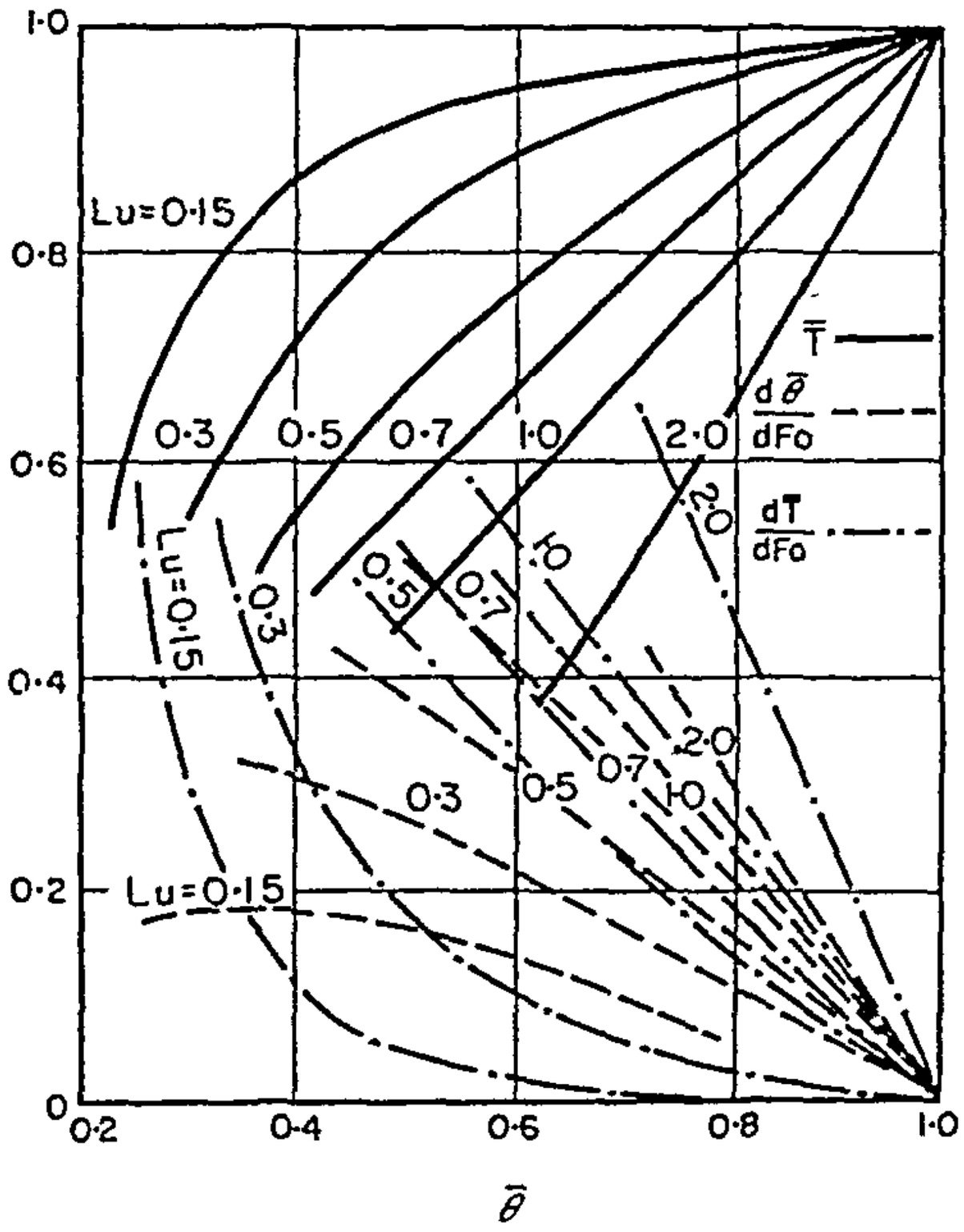


FIG. 25. Relation of heat and mass exchange characteristic to Lu

time the mean temperature of the material falls, but drying becomes more rapid; all the processes proceed more evenly in the whole period of the change of transfer potential of mass (Fig. 25).

Hence it is reasonable from the technological point of view to achieve conditions of heat and mass transfer (for example, drying) in which the value of the criterion Lu is as large as possible; this, along with the intensification of heating and drying, reduces the possibility of thermal decomposition and other undesirable consequences of high temperatures.

The criteria of surface and internal heat and mass exchange exert great influence on the course of the process. The criteria

of heat and mass exchange (Bi_q and Bi_m) characterize the rate of external transfer (heat by Bi_q and mass by Bi_m) in comparison with the rate of the internal transfer. At small values of the Biot criteria (Table 30) the rates of

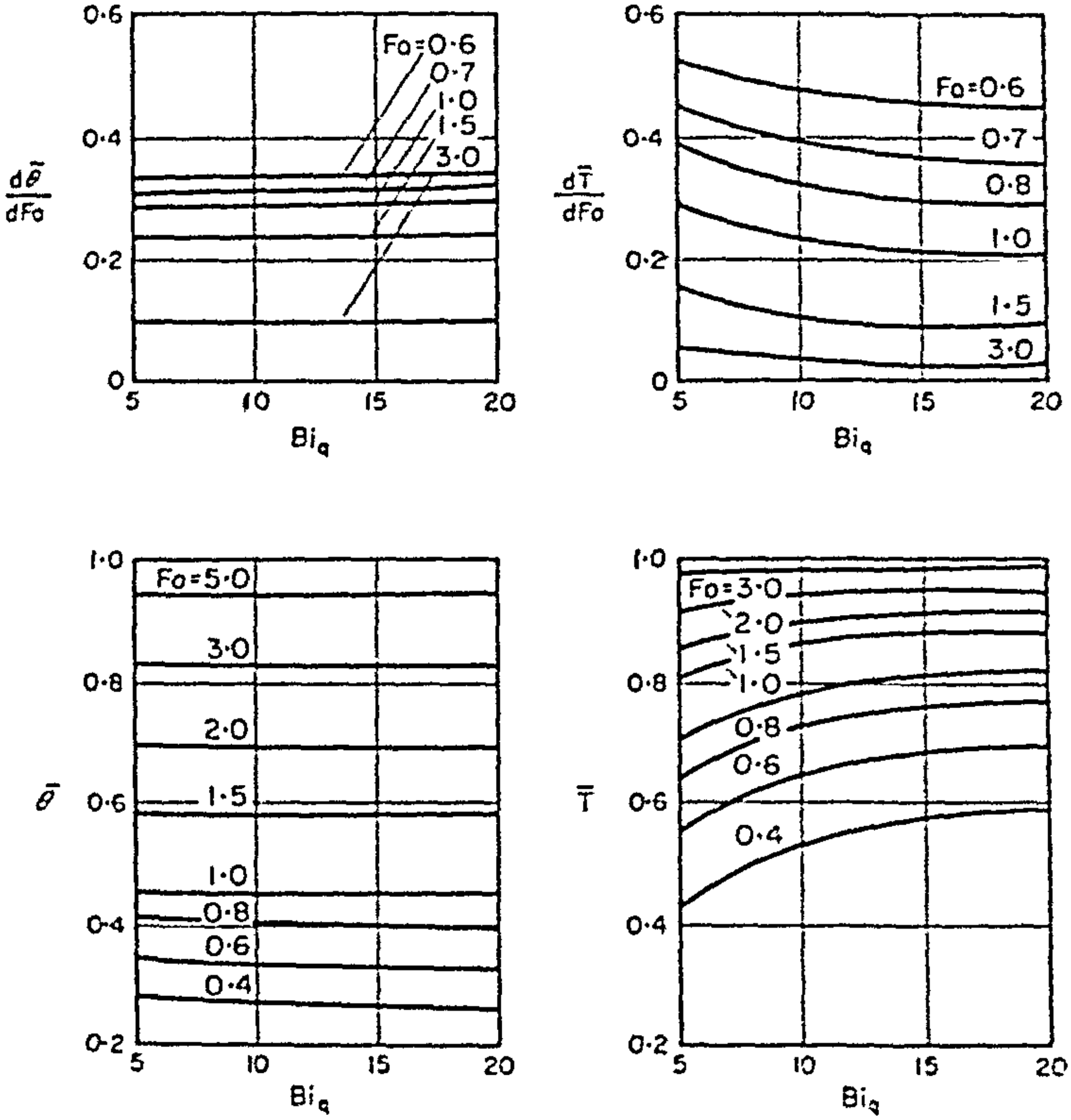


FIG. 26. Relation of the transfer potentials of heat and mass in an infinite plate to Bi_q

heating and drying are inconsiderable and the temperature and potential gradients are small. With increasing Bi_q and Bi_m the transfer process is intensified, and at the same time the transfer potential gradients increase. After the establishment of definite processes within the material (for a plate, Fo $\simeq$ 0·2, for a sphere, $\simeq$ 0·4) the Biot heat exchange criterion begins to affect only the thermal characteristics of heat and

TABLE 30. RELATIONS OF THE HEAT AND MASS

Index	Fo	$Bi_q = 0{\cdot}5$; $Bi_m = 0{\cdot}5$			
		$\frac{x}{R} = 0$	$\frac{x}{R} = 0{\cdot}25$	$\frac{x}{R} = 0{\cdot}50$	$\frac{x}{R} = 0{\cdot}75$
$T = \frac{t-t_0}{t_c-t_p}$	0·7	0·123	0·133	0·162	0·211
	0·8	0·147	0·157	0·185	0·232
	1·0	0·192	0·201	0·228	0·272
	1·5	0·293	0·301	0·324	0·362
	2·0	0·377	0·383	0·404	0·437
	3·0	0·507	0·513	0·528	0·553
	5·0	0·673	0·676	0·686	0·702
	7·0	0·768	0·771	0·777	0·788
	10·00	0·852	0·853	0·857	0·864
$\Theta = \frac{\theta_0-\theta}{\theta_0-\theta_p}$	0·7	—0·006	$0{\cdot}118 \cdot 10^{-1}$	$0{\cdot}640 \cdot 10^{-1}$	0·150
	0·8	$0{\cdot}59 \cdot 10^{-2}$	$0{\cdot}240 \cdot 10^{-1}$	$0{\cdot}753 \cdot 10^{-1}$	0·160
	1·0	$0{\cdot}303 \cdot 10^{-1}$	$0{\cdot}448 \cdot 10^{-1}$	$0{\cdot}974 \cdot 10^{-1}$	0·179
	1·5	$0{\cdot}889 \cdot 10^{-1}$	0·105	0·151	0·226
	2·0	0·144	0·159	0·201	0·271
	3·0	0·244	0·257	0·293	0·353
	5·0	0·411	0·420	0·447	0·492
	7·0	0·541	0·548	0·569	0·603
	10·0	0·684	0·689	0·703	0·726
$\frac{dT}{d\,\mathrm{Fo}}$	1·0	0·219	0·216	0·208	0·195
	1·5	0·183	0·181	0·174	0·163
	2·0	0·154	0·152	0·146	0·137
	3·0	0·110	0·109	0·105	$0{\cdot}987 \cdot 10^{-1}$
	5·0	$0{\cdot}611 \cdot 10^{-1}$	$0{\cdot}604 \cdot 10^{-1}$	$0{\cdot}584 \cdot 10^{-1}$	$0{\cdot}550 \cdot 10^{-1}$
$\frac{d\,\Theta}{d\,\mathrm{Fo}}$	1·0	0·121	0·118	0·109	$0{\cdot}955 \cdot 10^{-1}$
	1·5	0·113	0·111	0·103	$0{\cdot}911 \cdot 10^{-1}$
	2·0	0·107	0·104	$0{\cdot}975 \cdot 10^{-1}$	$0{\cdot}866 \cdot 10^{-1}$
	3·0	$0{\cdot}941 \cdot 10^{-1}$	$0{\cdot}922 \cdot 10^{-1}$	$0{\cdot}868 \cdot 10^{-1}$	$0{\cdot}779 \cdot 10^{-1}$
	5·0	$0{\cdot}734 \cdot 10^{-1}$	$0{\cdot}721 \cdot 10^{-1}$	$0{\cdot}683 \cdot 10^{-1}$	$0{\cdot}621 \cdot 10^{-1}$

TRANSFER CHARACTERISTICS TO THE BIOT CRITERIA

	$Bi_q = 10;\ Bi_m = 10$				
$\frac{x}{R} = 1{\cdot}00$	$\frac{x}{R} = 0$	$\frac{x}{R} = 0{\cdot}25$	$\frac{x}{R} = 0{\cdot}50$	$\frac{x}{R} = 0{\cdot}75$	$\frac{x}{R} = 1{\cdot}00$
0·276	0·567	0·594	0·670	0·785	0·924
0·295	0·619	0·643	0·709	0·810	0·932
0·332	0·699	0·717	0·769	0·848	0·944
0·413	0·814	0·825	0·857	0·904	0·963
0·481	0·872	0·880	0·901	0·934	0·973
0·588	0·931	0·935	0·947	0·964	0·985
0·723	0·978	0·979	0·983	0·988	0·995
0·803	0·993	0·993	0·994	0·996	0·998
0·873	0·999	0·999	0·999	0·999	1·000
0·265	$0{\cdot}714 \cdot 10^{-1}$	0·134	0·311	0·579	0·898
0·273	0·118	0·177	0·344	0·598	0·901
0·289	0·206	0·259	0·408	0·635	0·907
0·327	0·394	0·433	0·546	0·718	0·926
0·364	0·539	0·568	0·654	0·784	0·943
0·433	0·734	0·751	0·800	0·876	0·967
0·553	0·912	0·918	0·934	0·959	0·989
0·649	0·971	0·973	0·978	0·986	0·996
0·758	0·995	0·995	0·996	0·998	0·999
0·177	0·333	0·181	0·147	$0{\cdot}953 \cdot 10^{-1}$	$0{\cdot}515 \cdot 10^{-1}$
0·148	0·156	$0{\cdot}981 \cdot 10^{-1}$	$0{\cdot}801 \cdot 10^{-1}$	$0{\cdot}529 \cdot 10^{-1}$	$0{\cdot}272 \cdot 10^{-1}$
0·123	$0{\cdot}874 \cdot 10^{-1}$	$0{\cdot}619 \cdot 10^{-1}$	$0{\cdot}508 \cdot 10^{-1}$	$0{\cdot}339 \cdot 10^{-1}$	$0{\cdot}169 \cdot 10^{-1}$
$0{\cdot}901 \cdot 10^{-1}$	$0{\cdot}397 \cdot 10^{-1}$	$0{\cdot}313 \cdot 10^{-1}$	$0{\cdot}258 \cdot 10^{-1}$	$0{\cdot}174 \cdot 10^{-1}$	$0{\cdot}841 \cdot 10^{-2}$
$0{\cdot}505 \cdot 10^{-1}$	$0{\cdot}124 \cdot 10^{-1}$	$0{\cdot}101 \cdot 10^{-1}$	$0{\cdot}830 \cdot 10^{-2}$	$0{\cdot}561 \cdot 10^{-2}$	$0{\cdot}569 \cdot 10^{-2}$
$0{\cdot}777 \cdot 10^{-1}$	0·421	0·347	0·274	0·167	$0{\cdot}357 \cdot 10^{-1}$
$0{\cdot}751 \cdot 10^{-1}$	0·330	0·268	0·214	0·132	$0{\cdot}362 \cdot 10^{-1}$
$0{\cdot}723 \cdot 10^{-1}$	0·254	0·205	0·164	0·102	$0{\cdot}303 \cdot 10^{-1}$
$0{\cdot}661 \cdot 10^{-1}$	0·147	0·118	$0{\cdot}930 \cdot 10^{-1}$	$0{\cdot}592 \cdot 10^{-1}$	$0{\cdot}184 \cdot 10^{-1}$
$0{\cdot}539 \cdot 10^{-1}$	$0{\cdot}486 \cdot 10^{-1}$	$0{\cdot}391 \cdot 10^{-1}$	$0{\cdot}314 \cdot 10^{-2}$	$0{\cdot}196 \cdot 10^{-1}$	$0{\cdot}614 \cdot 10^{-2}$

mass transfer (T, $dT/d\mathrm{Fo}$) and does not influence the characteristics of mass transfer (Θ, $d\Theta/d\mathrm{Fo}$). Conversely, the mass exchange criterion of Biot begins to affect only the mass transfer characteristics (Figs. 26 and 27). For all calculated values of Bi_q the curve of the rate of drying proved to be the same (Fig. 28).

The criteria of surface heat and mass exchange, as follows from Fig. 24, increase for Bi_m and diminish for Bi_q with increase of $\bar{\Theta}$. The rate of change of their magnitudes is approx-

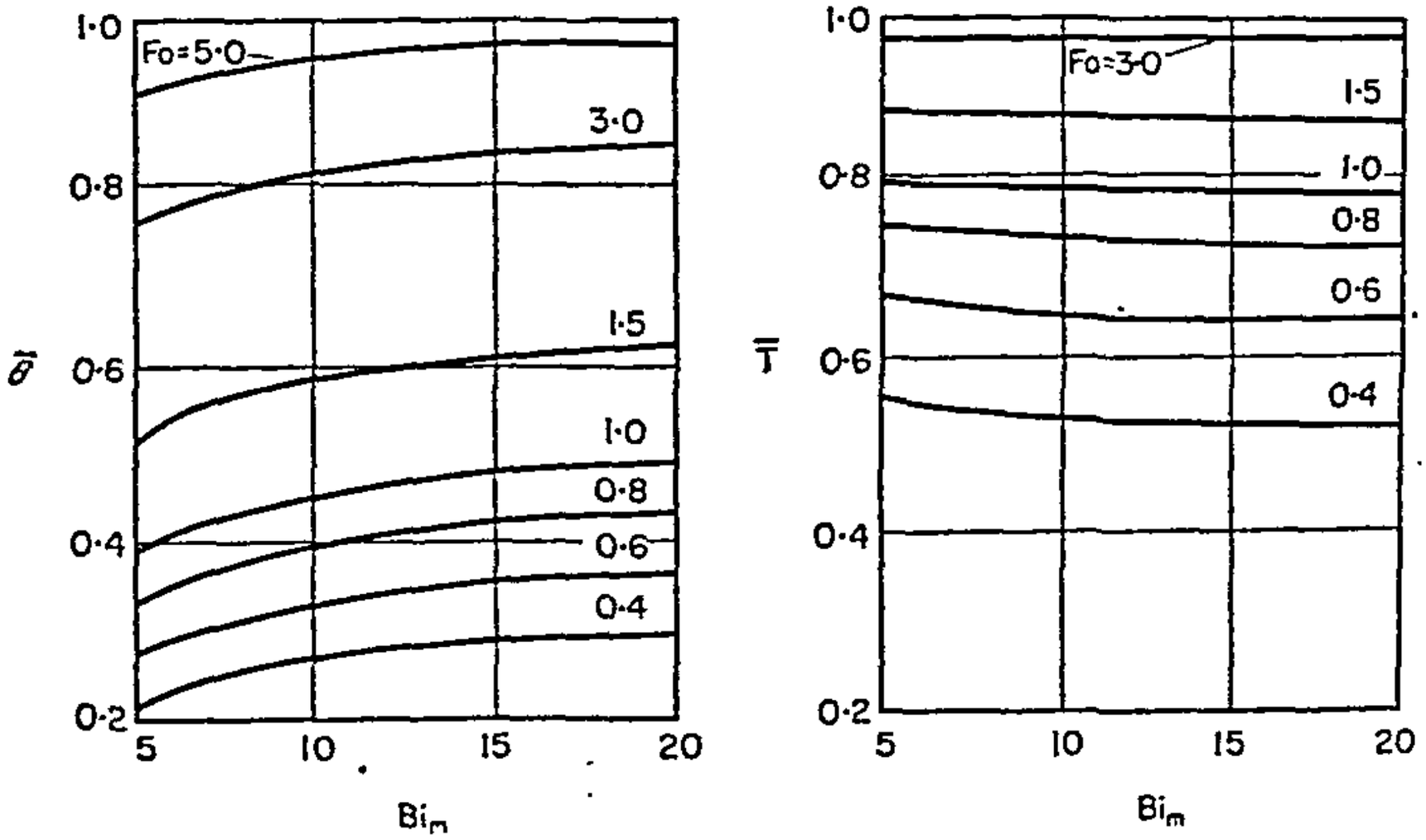


FIG. 27. The kinetics of heat and mass exchange in an infinite plate as it depends on Bi_m

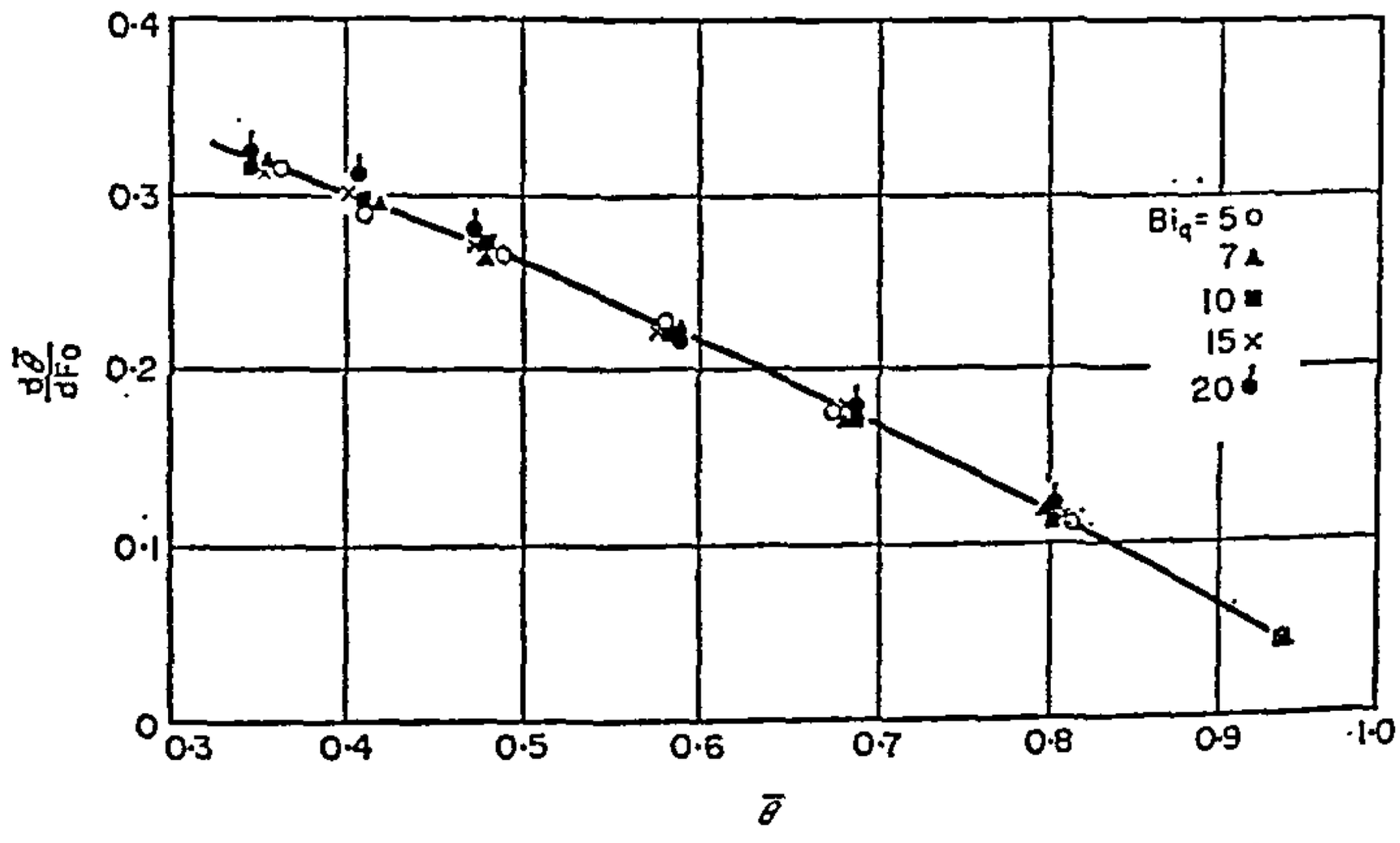


FIG. 28. Relation of the rate of drying to Bi_q

imately the same. The experimental data of P. D. Lebedev, and also of A. F. Chizhskii for Bi_m, confirm these results fairly well. With increasing mean non-dimensional temperature a reversed picture is observed: Bi_m diminishes and Bi_q increases with increasing $\bar{T}$. The heat and mass exchange criteria of Biot are not independent under actual conditions. In mathematical form their interconnection is expressed by the equations of the heat and mass balances. With the changing Biot heat exchange criterion the Biot mass exchange criterion also changes, since their numerical values have about the same order of magnitude. With the increase of both the Biot criteria, the temperature increases considerably, the rate of heating grows, and when $\bar{\theta}$ is small a stationary state is reached. Drying becomes more rapid and deeper.

The criteria of Kossovich (Ko) and Posnov (Pn) are also among the criteria of heat and mass transfer. In their peculiarities they recall the Biot criteria, but unlike the latter they characterize the internal transfers of heat and mass and have no definite influence on the processes in the external medium. The Kossovich criterion is the analogue of the heat exchange criterion of Biot, and the Posnov criterion is the analogue of Bi_m. The criterion Pn influences only the potential field of mass transfer, whilst Ko influences the temperature field (Fig. 30). With increasing Pn and Ko the non-dimensional (mean and local) transfer potentials of mass and temperature respectively diminish linearly. The linearity is strictly maintained for all values of Fo. With the diminution of the criterion Pn along with increase in the rate of drying, the temperature conditions improve, since the process goes on at lower temperatures of the material. Calculation shows that as the moisture content diminishes, there is a sharp fall in the criterion Pn (see Fig. 29). This behaviour of Pn is confirmed by experiment. The Kossovich criterion increases with increasing $\bar{\theta}$ and falling $\bar{T}$. The relation between the criteria of heat and mass transfer to $\bar{T}$ and $\bar{\theta}$ differs diametrically from

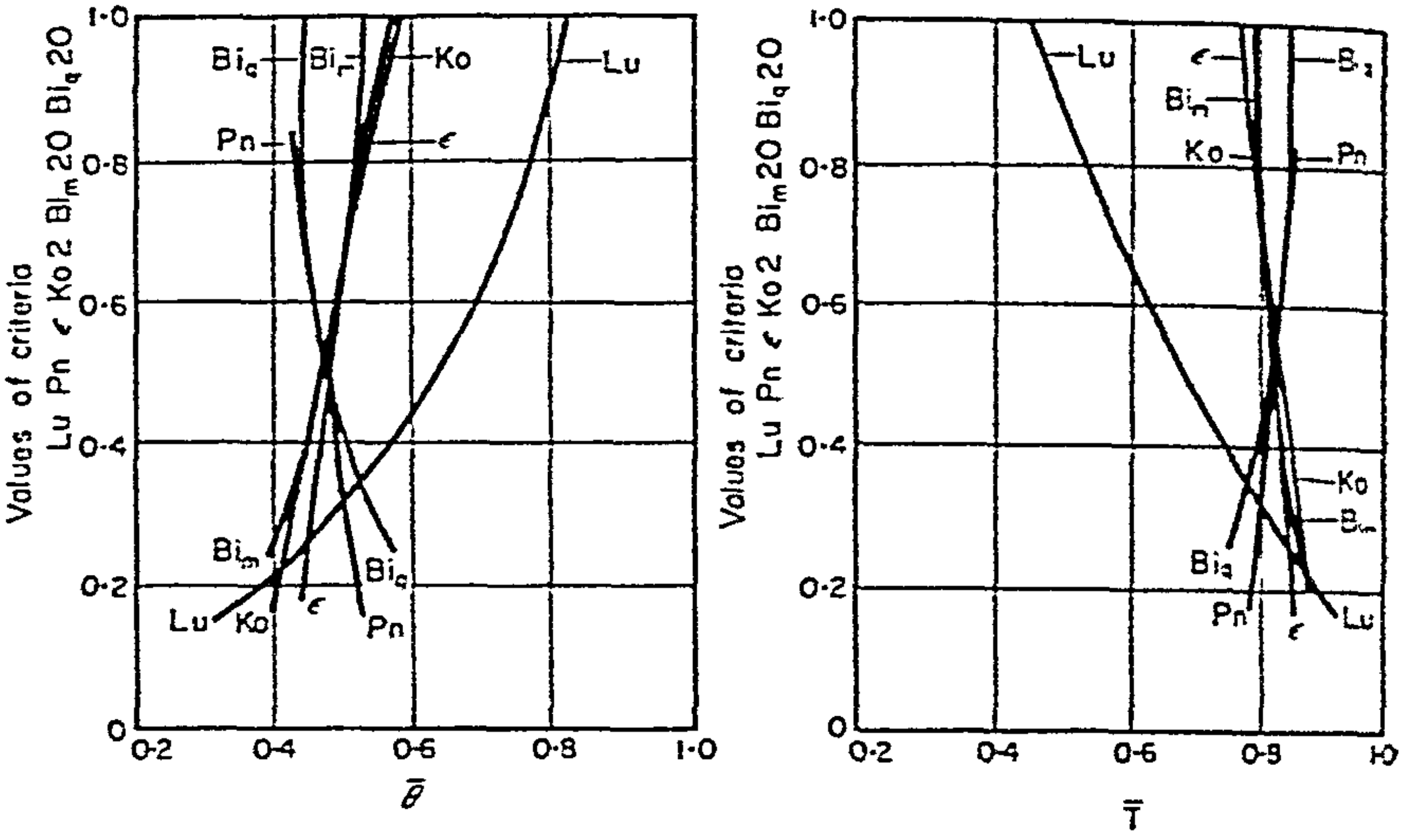


FIG. 29. Influence of $\bar{T}$ and $\bar{\Theta}$ on the criteria of heat and mass exchange

the relation of the criteria of heat and mass exchange to the same quantities.

The nature of the relation revealed between the heat and mass transfer and the criteria Bi_m and Pn enables us to explain the method of treating the experimental data established by Lebedev on the rate of drying.

He showed that for any method of drying $d\bar{\Theta}/d\mathrm{Fo}$ was uniquely determined by two parameters: the ratio Bi_m/Pn and the amount of moisture removed $(1 - \bar{u})$. It follows from the above analysis that the surface mass exchange depends on the criterion Bi_m, and the internal mass exchange on Pn. Hence the combination of the criteria Bi_m and Pn completely determines the mass exchange in the body.

This means that Θ and $d\Theta/d\mathrm{Fo}$, as the quantities which characterize the mass exchange processes, must be completely described by the combined criteria Bi_m and Pn. This relation must be applicable to the different methods of drying, since the mechanism of heat and mass transfer is the same for all of them.

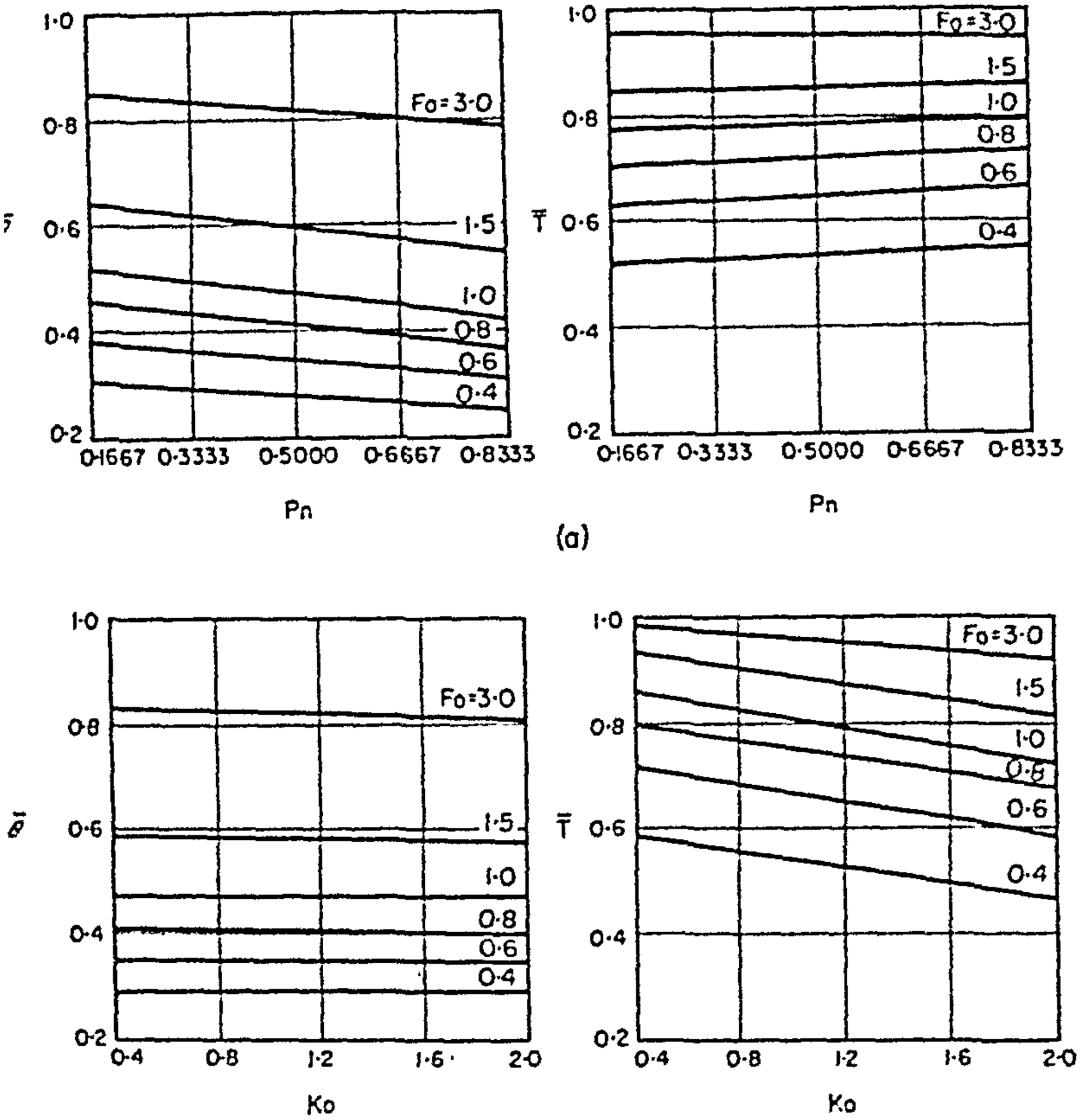


FIG. 30. Relation of the transfer potentials (a) to Pn, and (b) to Ko

On the basis of this analysis of the influence of the criteria of similarity on the heat and mass exchange, we conclude that the combined criteria Bi_q and Ko, i. e. Bi_q/Ko must play an analogous part in the grouping of experimental data on the rate of heating of moist material. This follows from the fact that the criteria Bi_q and Ko are both mainly concerned with the heat exchange processes, both on the surface of the material and in its interior.

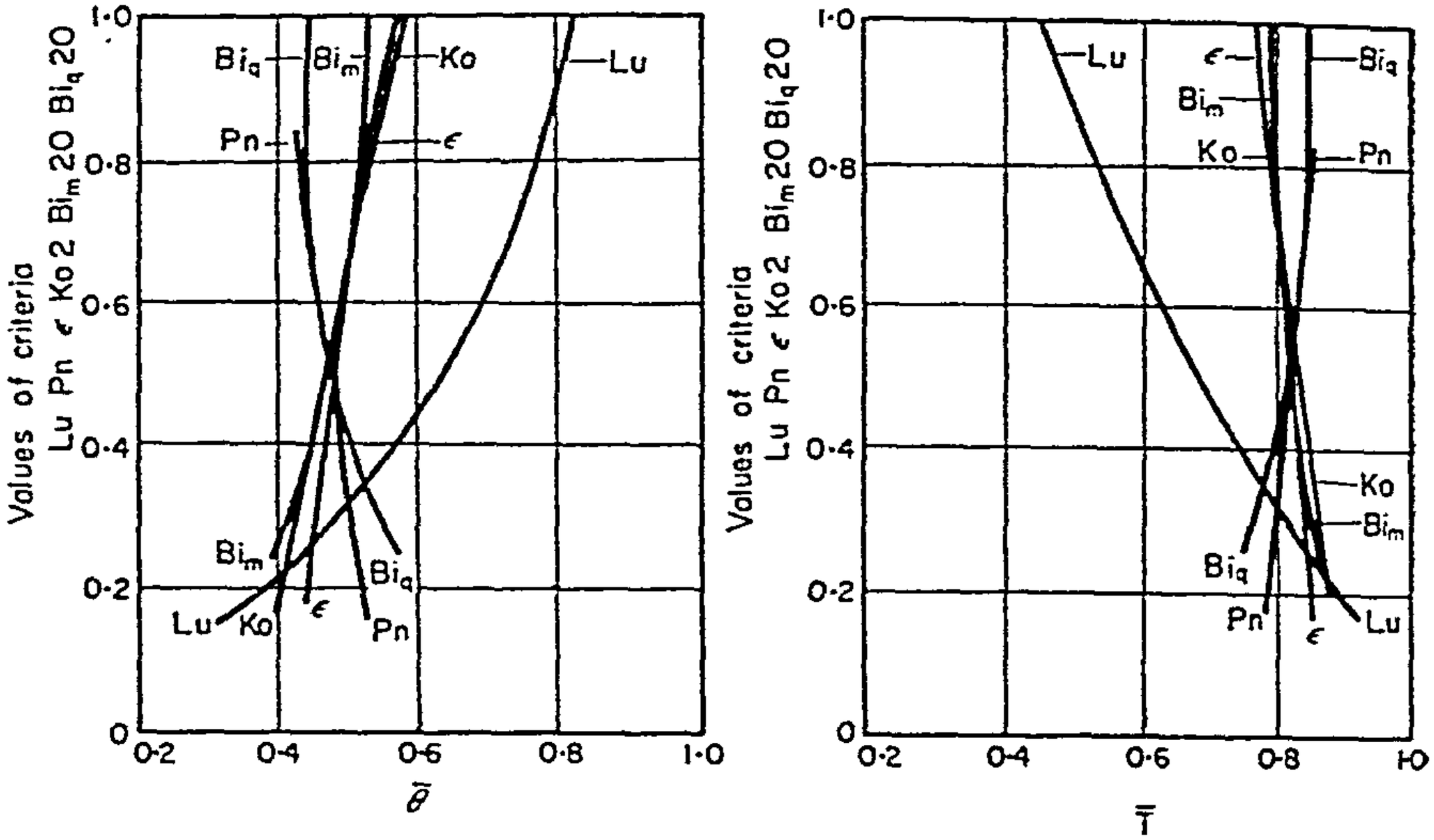


FIG. 29. Influence of $\bar{T}$ and $\bar{\Theta}$ on the criteria of heat and mass exchange

the relation of the criteria of heat and mass exchange to the same quantities.

The nature of the relation revealed between the heat and mass transfer and the criteria Bi_m and Pn enables us to explain the method of treating the experimental data established by Lebedev on the rate of drying.

He showed that for any method of drying $d\bar{\Theta}/d\mathrm{Fo}$ was uniquely determined by two parameters: the ratio Bi_m/Pn and the amount of moisture removed $(1 - \bar{u})$. It follows from the above analysis that the surface mass exchange depends on the criterion Bi_m, and the internal mass exchange on Pn. Hence the combination of the criteria Bi_m and Pn completely determines the mass exchange in the body.

This means that Θ and $d\Theta/d\mathrm{Fo}$, as the quantities which characterize the mass exchange processes, must be completely described by the combined criteria Bi_m and Pn. This relation must be applicable to the different methods of drying, since the mechanism of heat and mass transfer is the same for all

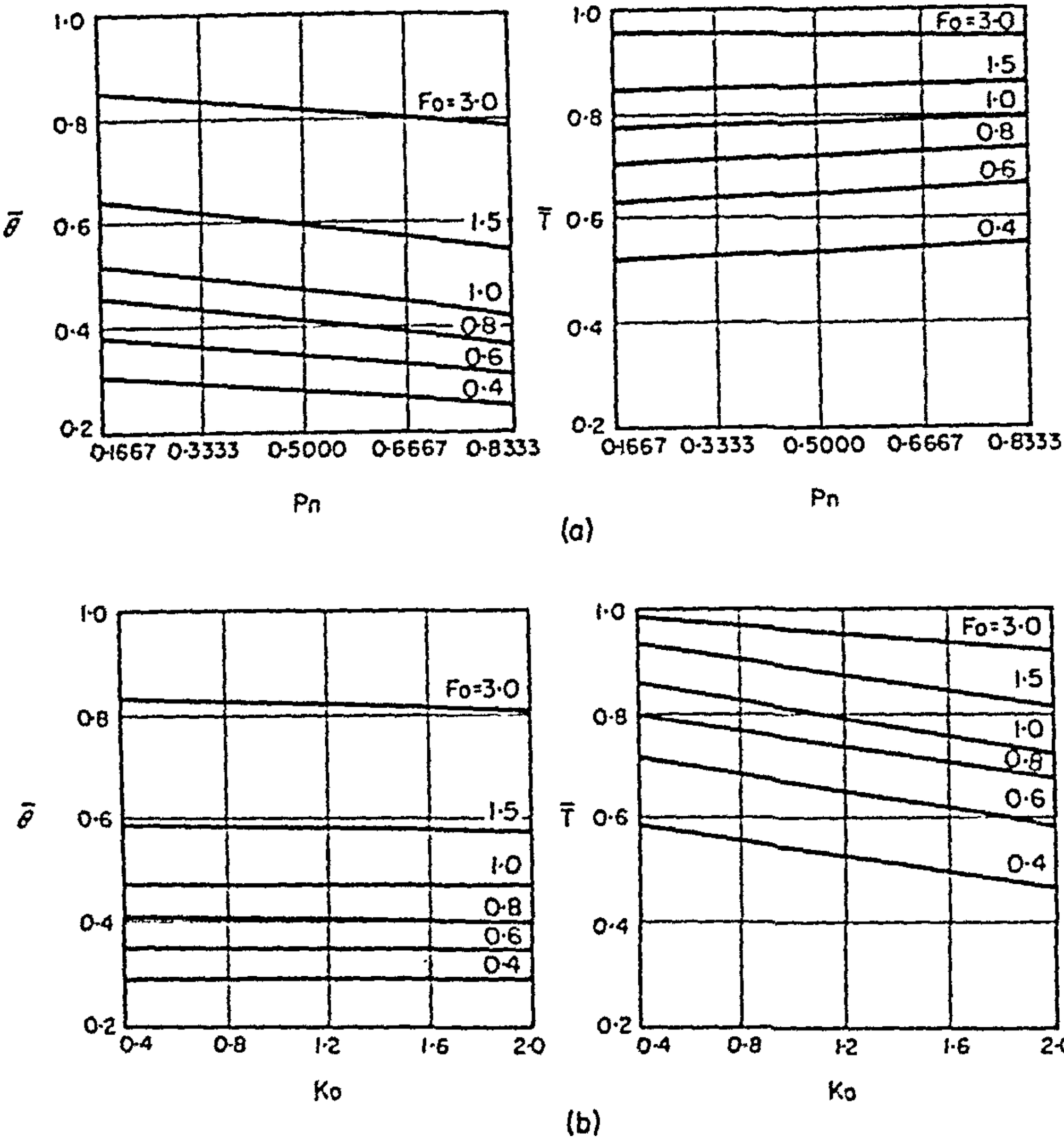


Fig. 30. Relation of the transfer potentials (a) to Pn, and (b) to Ko

On the basis of this analysis of the influence of the criteria of similarity on the heat and mass exchange, we conclude that the combined criteria Bi_q and Ko, i. e. Bi_q/Ko must play an analogous part in the grouping of experimental data on the rate of heating of moist material. This follows from the fact that the criteria Bi_q and Ko are both mainly concerned with the heat exchange processes, both on the surface of the material and in its interior.

The phase change criterion ε exercises the same influence on the process of heat and mass transfer as the Kossovich criterion. An exception is the intensity of the action on the thermal characteristics: the influence of ε is less than that of Ko. The course of the graphical representations of the heat and mass exchange characteristics of ε and Ko, as well as an analysis of the general solutions, indicates the existence of a functional relation between the two criteria.

In connection with the analysis of the criterion ε, it is of interest to consider the influence of the phase change term $(c'_T\ \varepsilon\varrho/c_q)\partial\theta/\partial\tau$ (see equation (I)) on the distribution of the transfer potentials of heat and mass, and also on their rate of change. It is possible to estimate the influence of this term analytically by comparing the results of the calculation of the system (I – VI) with positive and negative signs to the term $(c'_T\ \varepsilon\varrho/c_q)\ \partial\theta/\partial\tau$. The difference between the results will be double the value of the term of phase transformation. Calculation shows that the influence of double the phase-change term is:

TABLE 31. CRITERION ARITHMETIC MEAN INFLUENCE %

Dependance on Criterion	$\bar{T}$	$\bar{\theta}$	$\frac{d\bar{T}}{d\,\mathrm{Fo}}$	$\frac{d\,\bar{\theta}}{d\,\mathrm{Fo}}$
Lu	20 (13÷25	4 (1÷6)	2 (—4÷+7)	10 (3—16)
Bi_q	15 (12÷21)	4 (4)	5 (2—7)	9
Bi_m	20 (19÷11)	4 (3÷6)	8 (4÷12)	9 (8÷11)
Pn	21 (24÷18)	4 (0÷7)	6 (—8÷22)	11 (5÷16)
Ko	21 (6÷36)	3 (—2—+8)	6 (1—13)	12 (8÷16)

Hence the most important influence of this term is on the temperature field (in general the influence is about 10 per cent), but there is hardly any influence on the potential field of mass (2 per cent) or the rate of heating of the material (3 per cent). The contribution of the internal phase change to the non-dimensional rate of drying is also very small and amounts to 5 per cent.

If the heat and mass transfer proceeds in a medium with variable temperature, the criterion Pd appears in the general solution. This criterion characterizes the rate of change of the temperature of the medium. We will make a brief analysis of the influence of the criterion Pd on the heat and mass transfer when the temperature of the heat carrier falls with the time according to a linear law.

The non-dimensional temperature first of all increases with increasing Fo, reaches a maximum and then begins to fall (Fig. 31). The greater the value of Pd, the earlier (that

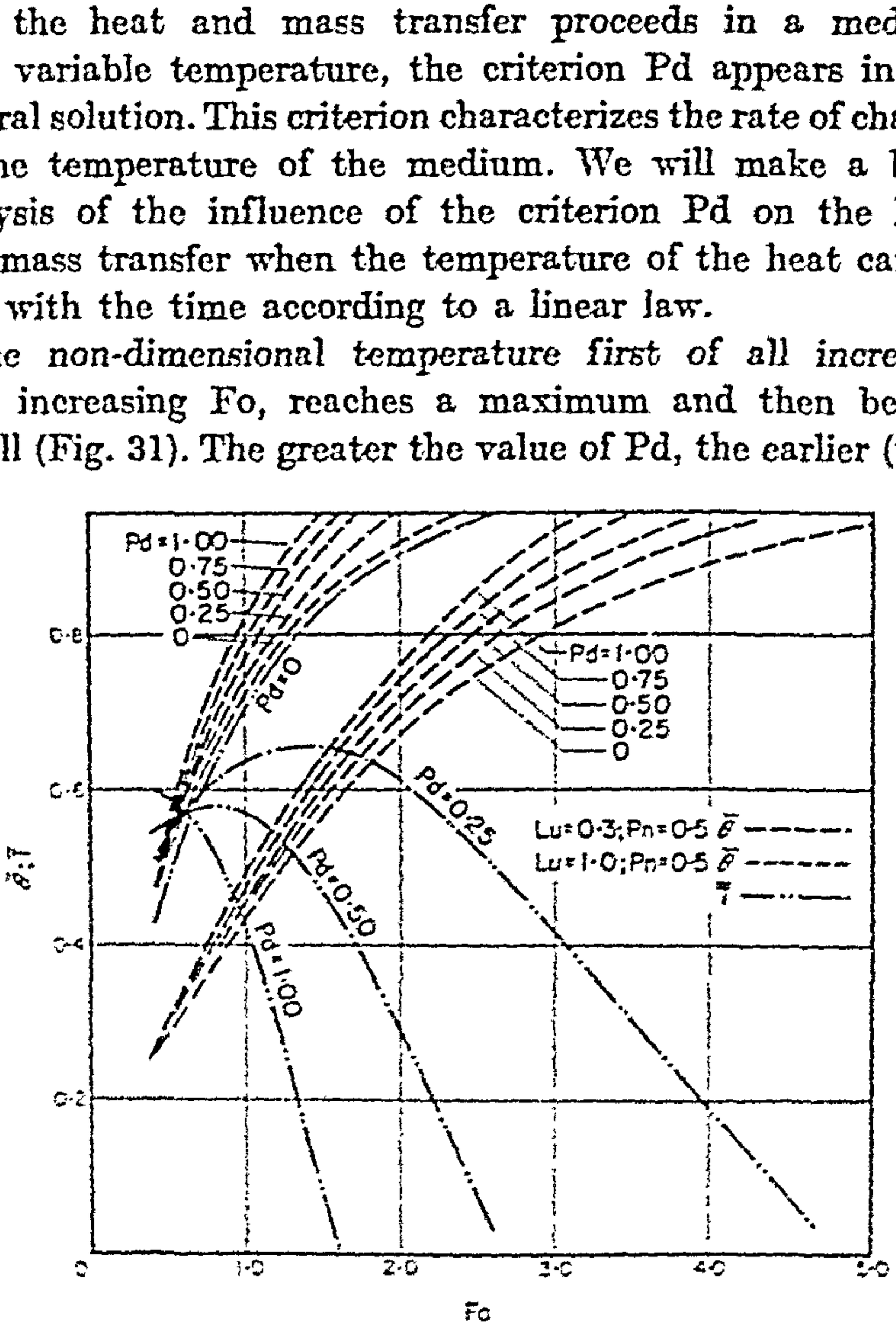


Fig. 31. Relation of the transfer potentials to Pd and Fo

is, for the lower Fo and $\bar{\theta}$) does the maximum of the temperature of the material occur.

Up to $T = T_{max}$ the temperature on the surface rises considerable more rapidly than in the centre of the material. After passing through T_{max} the surface layer of the body begins to cool, since at this moment the temperature of the medium falls below that of the material. However, the heating up of the centre continues, since $t_w > t_c$. As soon as $t_c \approx t_w$ all layers of the body are included in the cooling. A knowledge of the moment when the maximum temperature is attained and of its absolute value is important for the correct organization of the drying process. In order to prevent the condensation of the heat carrier on the material at about the instant $Fo_{T_{max}}$ the drying process should be suspended.

With increasing Pd the rate of heating of the material diminishes according to a linear law. The change of the transfer potential of mass and the rate of drying with the criterion Pd is small: with the growth of Pd and Fo they increase. Thus, for Fo = 0·4 the increase in the rate of drying from Pd = 0·25 to Pd = 1·0 was 8·5 per cent, and the change in the potential of mass ($\bar{\theta}$) was 3·5 per cent. For Fo = 0·8 we have respectively 11·8 per cent and 5·3 per cent. With increasing Lu the same relationship of $\bar{\theta}$ to Pd is preserved, but is found at higher values of $\bar{\theta}$.

The influence of the parametric criteria of non-uniformity of the initial distribution of temperature and transfer potential of mass on heat and mass transfer is apparent only up to a certain value of Fo. For the degree of accuracy necessary in practice it is permissible to neglect the influence of the parametric criterion W on the mass exchange characteristics of heat and mass transfer (beginning from Fo $\geqslant$ 1·0). The influence of the criterion V on the distribution of transfer potentials is very considerable, and hence it must be taken into account in working up experimental data up to $\bar{\theta}_1$ = 0·7 and Fo = 2·5.

Finally let us consider the influence of the form of the body on the course of heat and mass transfer. It will be seen from Fig. 32 that with the increase of the form factor the mean temperature of the body falls for one and the same mass

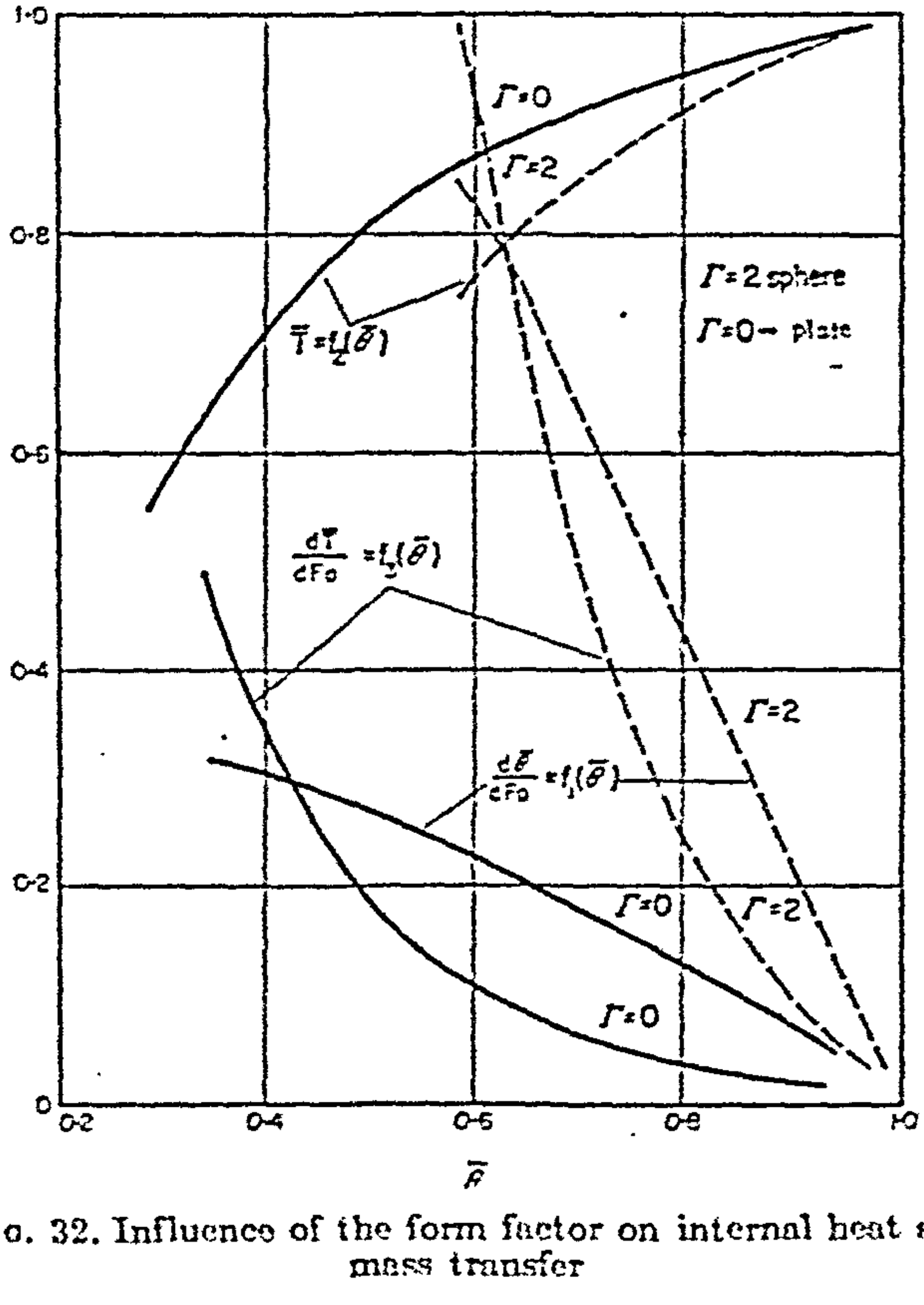


FIG. 32. Influence of the form factor on internal heat and mass transfer

content. The rate of drying in all sections of the process is higher for a sphere than for an infinite plate. The fall in the rates of heating and drying with increasing $\bar{\theta}$ is also greater in bodies with a higher value of Γ. Thus, an increase in Γ considerably accelerates the drying process and lowers the mean temperature of the material. Hence it is more rational to dry a moist body in the form of a sphere than as a cylinder,

and bodies of cylindrical form dry better than those in the form of a plate.

Owing to the character of the influence of the individual criteria on the heat and mass exchange processes it becomes possible, in describing the heat and mass transfer, to neglect some of the criteria and thus considerably simplify the model considered. Thus the simplified criterial equations for the mean characteristics can be written in the following form:

$$\overline{T} = \overline{T}\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_q}{\mathrm{Ko}}, \varepsilon, \mathrm{Pd}, V, W\right),$$

$$\overline{\Theta} = \overline{\Theta}\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_m}{\mathrm{Pn}}, \mathrm{Pd}, V\right),$$

$$\frac{d\overline{T}}{d\,\mathrm{Fo}} = \varphi_1\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_q}{\mathrm{Ko}}, \varepsilon, \mathrm{Pd}, V, W\right),$$

$$\frac{d\overline{\Theta}}{d\,\mathrm{Fo}} = \varphi_2\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_m}{\mathrm{Pn}}, \mathrm{Pd}, V\right).$$

If the temperature of the medium is constant with respect to the time, then $\mathrm{Pd} = 0$. With a constant initial distribution of temperature and transfer potential of matter, $V = 0$ and $W = 0$. For a stationary process of heat and mass transfer, $\mathrm{Fo} = \infty$.

§ 7. Heat and Mass Transfer in the Presence of Phase and Chemical Transformation

In the thermal treatment of a number of moist materials under pressure, and in drying these materials at high temperature, along with phase-change of the associated matter there is also thermal destruction of the skeletal material. It is therefore of interest to consider heat and mass transfer when accompanied by change in the dry mass of the material.

The system of differential equations of molecular heat and mass transfer accompanied by thermal decomposition of the

dry mass, with zonal calculation, can be written in the following form:

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \varepsilon \frac{\varrho}{C_q} \frac{\partial \omega_m}{\partial \tau} + \frac{Q_d}{C_q} kf(\omega_{1d}), \qquad (7.1)$$

$$\frac{\partial \omega_m}{\partial \tau} = a_m \nabla^2 \omega_m + a_m \delta_m \nabla^2 t, \qquad (7.2)$$

$$\frac{\partial \omega_{d_2}}{\partial \tau} = a_{d_2} \nabla^2 \omega_{d_2} + a_{d_2} \delta_{d_2} \nabla^2 t - \frac{\partial \omega_{d_1}}{\partial \tau}, \qquad (7.3)$$

$$\frac{\partial \omega_{d_1}}{\partial \tau} = - kf(\omega_{d_1}), \qquad (7.4)$$

where ω is the concentration of matter in unit volume of the material, K is the total velocity constant of the reaction, the subscript m denotes associated matter, d_1 denotes the solid phase of the skeleton material, and d_2 denotes the gaseous products of the decomposition of the material. The other symbols remain as before.

In distinction from the cases already considered, the system of differential equations is written for the sake of clarity not in symbols denoting the transfer potentials of mass, but in symbols denoting the concentrations. The first term on the righthand side of the equation of heat transfer (7.1) denotes the change in temperature produced by molecular transfer, the second denotes that produced by the phase transformation of associated matter, while the third denotes that produced by the thermal destruction of the material. If $Q_d < 0$ the reaction is endothermic and proceeds with the absorption of heat; when $Q_d > 0$ the reaction is exothermic and heat is evolved in the process of heat and mass transfer. Equations (7.2) and (7.3) are respectively the differential equations of mass transfer of associated matter and of the gaseous products of thermal decomposition. The last term of (7.3), $\partial \omega_{d1}/\partial \tau$, denotes the rate of formation of gaseous products as the result of the decomposition of the solid phase of the skeleton. The rate of the reaction of thermal decomposition depends on the concentration of the reacting components and products of decomposition. To a first approx-

imation it is permissible to consider it a function of the concentration ω_{d1}. This approximation is reflected in equation (7.4). If the reaction is of the first order, equation (7.4) will assume the form

$$\frac{\partial \omega_{d1}}{\partial \tau} = -k\,\omega_{d1}\,.$$

Boundary conditions of the third kind for the system of equations (7.1)–(7.4) can be written thus:

$$-\lambda_q \nabla t_W + \alpha(t_c - t_W) - (1-\varepsilon)\,\varrho\beta_m(\omega_{mW} - \omega_m^P) = 0, \quad (7.5)$$

$$a_m \nabla \omega_{mW} + a_m \delta_m \nabla t_W + \beta_m(\omega_{mW} - \omega_m^P) = 0\,, \quad (7.6)$$

$$a_{d2} \nabla \omega_{d2W} + a_{d2}\,\delta_{d2} \nabla t_W + \beta_{d2}(\omega_{d2W} - \omega_{d2}^P) = 0\,. \quad (7.7)$$

From the system of equations (7.1)–(7.4) and the boundary conditions (7.5)–(7.7) we can obtain new criteria of similarity and simplexes, but the form of the majority of the criteria used above is changed. We have:

the inertial criterion of the field of bound matter and of the products of decomposition in relation to the temperature field:

$$\mathrm{Lu}_m = \frac{a_m}{a_q} \quad \text{and} \quad \mathrm{Lu}_d = \frac{a_{d2}}{a_q}\,;$$

the Posnov criterion for bound matter and the gasiform products of decomposition:

$$\mathrm{Pn}_m = \frac{\delta_m(t_a - t_0)}{\omega_m^0 - \omega_m^P} \quad \text{and} \quad \mathrm{Pn}_d = \frac{\delta_{d2}(t_a - t_0)}{\omega_{d2}^0}\,;$$

the Kossovich criterion for bound matter and the process of thermal decomposition:

$$\mathrm{Ko}_m = \frac{\varrho(\omega_m^0 - \omega_m^P)}{C_q(t_c - t_0)} \quad \text{and} \quad \mathrm{Ko}_d = \frac{Q_d}{C_q}\,\frac{\omega_{d1}^0}{t_c - t_0'}\,;$$

the Biot mass exchange criterion for bound matter and the products of decomposition:

$$\mathrm{Bi}_m = \beta_m R/a_m \quad \text{and} \quad \mathrm{Bi}_d = \beta_d R/a_{d2}\,;$$

the Hess criterion, analogous to the Arrhenius criterion used in chemical kinetics:

$$\mathrm{Ge} = (kR^2/a_{d2})(\omega_{d1}^0)^{n-1}$$

where n is the order of the chemical reaction, if $f(\omega_{d1}) = \omega_{d1}^n$; the non-dimensional simplex of decomposition of the dry mass $W_0 = \omega_{d2}^0/\omega_{d1}^0$, and also the non-dimensional simplexes of the temperature T, the concentration of bound matter U, the concentration W_1 of the solid phase of the skeleton, and W_2 of the products of its decomposition are

$$T = \frac{t - t_0}{t_c - t_0}, \quad U = \frac{\omega_m^0 - \omega_m}{\omega_m^0 - \omega_m^P}, \quad W_1 = \frac{\omega_{d1}^0 - \omega_{d1}}{\omega_{d1}^0}, \quad W_2 \frac{\omega_{d2} - \omega_{d2}^0}{\omega_{d2}^0}.$$

The remaining non-dimensional complexes are written as before. The index 0 denotes the initial condition, the index P denotes the equilibrium condition, and the index W the value of the parameter on the surface of the body.

The method of solving this system of equations of heat and mass transfer in the presence of phase and chemical transformations does not differ in principle from the method of solution if the problems considered above. Omitting the intermediate calculations, we append the final results of the solution for a symmetrical problem with constant initial distribution of concentration and in conditions where chemical reaction is of the first order

(a) *Solutions for an infinite plate*

$$T(X, \mathrm{Fo}) = 1 -$$

$$- \mathrm{Ko}_d \left(1 + \frac{\tilde{P}_2 \cos[\nu_1 \sqrt{(\mathrm{Lu}_d \mathrm{Ge})}\, X] - \tilde{P}_1 \cos[\nu_2 \sqrt{(\mathrm{Lu}_d \mathrm{Ge})}\, X]}{\tilde{P}_1 \tilde{Q}_2 - \tilde{P}_2 \tilde{Q}_1}\right) \times$$

$$\times \exp(-\mathrm{Lu}_d \mathrm{Ge}\, \mathrm{Fo}) - \sum_{n=1}^{\infty} \sum_{i=1}^{2} C_{ni} (\cos \nu_i \cdot \mu_n X \cdot$$

$$\cdot \exp(-\mu_n^2 \mathrm{Fo}), \qquad (7.8)$$

$$U(X, \mathrm{Fo}) = 1 + \frac{\mathrm{Ko}_d}{\varepsilon \mathrm{Ko}_m} \times$$

$$\times \frac{\widetilde{P}_2(1-\nu_1^2)\cos[\nu_1\sqrt{(\mathrm{Lu}_d\,\mathrm{Ge})}\,X] - \widetilde{P}_1(1-\nu_2^2)\cos[\nu_2\sqrt{(\mathrm{Lu}_d\,\mathrm{Ge})}\,X]}{\widetilde{P}_1\widetilde{Q}_2 - \widetilde{P}_2\widetilde{Q}_1} \times$$

$$\times \exp(-\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Fo}) + \frac{1}{\varepsilon\,\mathrm{Ko}_m}\sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}(1-\nu_i^2)\cos\nu_i \cdot$$

$$\cdot \mu_n X \cdot \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (7.9)$$

$$W_1(X, \mathrm{Fo}) = 1 - \exp(-\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Fo}), \qquad (7.10)$$

$$W_2(X, \mathrm{Fo}) = -1 + \left(\frac{1}{W_0} + \mathrm{Ko}_d\,\mathrm{Pn}_d\,\frac{\widetilde{P}_2\widetilde{D}_1 - \widetilde{P}_1\widetilde{D}_2}{\widetilde{P}_1\widetilde{Q}_2 - \widetilde{P}_2\widetilde{Q}_1}\right)\times$$

$$\times \frac{\cos\sqrt{(\mathrm{Ge})}\,X}{\cos\sqrt{(\mathrm{Ge})} - \frac{1}{\mathrm{Bi}_d}\sqrt{(\mathrm{Ge})}\sin\sqrt{\mathrm{Ge}}} \cdot \exp(-\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Fo}) -$$

$$-\left[\frac{1}{W_0} + \mathrm{Ko}_d\,\mathrm{Pn}_d\,\frac{\widetilde{P}_2\frac{\nu_1^2}{1/\mathrm{Lu}_d - \nu_1^2}\cos[\nu_1\sqrt{(\mathrm{Lu}_d\,\mathrm{Ge})}\,X]}{\widetilde{P}_1\widetilde{Q}_2 - \widetilde{P}_2\widetilde{Q}_1} -\right.$$

$$\left. - \frac{\widetilde{P}_1\frac{\nu_2^2}{1/\mathrm{Lu}_d - \nu_2^2}\cos[\nu_2\sqrt{(\mathrm{Lu}_d\mathrm{Ge})}\,X]}{\widetilde{P}_1\widetilde{Q}_2 - \widetilde{P}_2\widetilde{Q}_1}\right]\exp(-\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Fo}) +$$

$$+ \sum_{m=1}^{\infty} A_m\left[\frac{1}{\mathrm{Lu}_d} + \frac{1}{W_0}\,\frac{\mathrm{Ge}}{\mu_m^2(\mu_m^2 - \mathrm{Lu}_d\,\mathrm{Ge})} +\right.$$

$$\left. + \mathrm{Pn}_d\sum_{i=1}^{2} M_{mi}D_{mi}\right]\cos\frac{\mu_m}{\sqrt{\mathrm{Lu}_d}}X \cdot \exp(-\mu_m^2\,\mathrm{Fo}) +$$

$$+ \mathrm{Pn}_d\sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}\left[D_{ni}\,\frac{\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}_d}}X}{\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}_d}} - \frac{1}{\mathrm{Bi}_d}\,\frac{\mu_n}{\sqrt{\mathrm{Lu}_d}}\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}_d}}} -\right.$$

$$\left. - \frac{\nu_i^2}{\frac{1}{\mathrm{Lu}_d} - \nu_i^2}\cos\nu_i\,\mu_n X\right]\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (7.11)$$

Here

$$C_{n1} = 2\,\frac{\left(\varepsilon\,\mathrm{Ko}_m K_1 - 1 - \frac{\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Ko}_d}{\mu_n^2 - \mathrm{Lu}_d\mathrm{Ge}}\right)P_{n2} - \varepsilon\,\mathrm{Ko}_m Q_{n2}}{\mu_n\varphi_n},$$

$$C_{n2} = -2\frac{\left(\varepsilon\,\mathrm{Ko}_m K_1 - 1 - \dfrac{\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Ko}_d}{\mu_n^2 - \mathrm{Lu}_d\,\mathrm{Ge}}\right) P_m - \varepsilon\,\mathrm{Ko}_m Q_{n1}}{\mu_n \psi_n},$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nl} = \left[1 + \frac{1}{\mathrm{Bi}_q} + K_1(1 - \nu_l^2)\right] \sin \nu_l \mu_n + \frac{1}{\mathrm{Bi}_q} \nu_l \mu_n \cos \nu_l \mu_n,$$

$$B_{nl} = (1 - \nu_l^2) \sin \nu_l \mu_n + \\ + \frac{(1 - \nu_l^2) + \varepsilon\,\mathrm{Ko}_m \mathrm{Pn}_m}{\mathrm{Bi}_m} (\sin \nu_l \mu_n + \nu_l \mu_n \cos \nu_l \mu_n),$$

$$P_{nl} = (1 - \nu_l^2) \cos \nu_l \mu_n - \frac{(1 - \nu_l^2) + \varepsilon\,\mathrm{Ko}_m \mathrm{Pn}_m}{\mathrm{Bi}_m} \nu_l \mu_n \sin \nu_l \mu_n,$$

$$Q_{nl} = [1 + K_1(1 - \nu_l^2)] \cos \nu_l \mu_n - \frac{1}{\mathrm{Bi}_q} \nu_l \mu_n \sin \nu_l \mu_n,$$

$$D_{nl} = \left(\cos \nu_l \mu_n - \frac{1}{\mathrm{Bi}_l} \nu_l \mu_n \sin \nu_l \mu_n\right) \frac{\nu_l^2}{\dfrac{1}{\mathrm{Lu}_d} - \nu_l^2} - \\ - \frac{1}{\mathrm{Bi}_d} \nu_l \mu_n \sin \nu_l \mu_n.$$

P_{ml}, Q_{ml}, D_{ml} and $\tilde{P}_l$, $\tilde{Q}_l$, $\tilde{D}_l$ are written in the same form as P_{nl}, Q_{nl} and D_{nl}, the characteristic root μ_n being replaced by μ_m and $\sqrt{\mathrm{Lu}_d\,\mathrm{Ge}}$ respectively.

Moreover,

$$\nu_i^2 = \frac{1}{2}\left\{\left(1 + \varepsilon\,\mathrm{Ko}_m \mathrm{Pn}_m + \frac{1}{\mathrm{Lu}_m}\right) + \right.$$

$$\left. + (-1)^i \sqrt{\left[\left(1 + \varepsilon\,\mathrm{Ko}_m \mathrm{Pn}_m + \frac{1}{\mathrm{Lu}_m}\right)^2 - \frac{4}{\mathrm{Lu}_m}\right]}\right\},$$

$$(i = 1, 2)$$

$$M_{m1} = \frac{\left(\varepsilon\,\mathrm{Ko}_m K_1 - 1 - \dfrac{\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Ko}_d}{\mu_m^2 - \mathrm{Lu}_d\,\mathrm{Ge}}\right) P_{m2} - \varepsilon\,\mathrm{Ko}_m Q_{m2}}{P_{m1} Q_{m2} - P_{m2} Q_{m1}},$$

$$M_{m2} = -\frac{\left(\varepsilon\,\mathrm{Ko}_m K_1 - 1 - \dfrac{\mathrm{Lu}_d\,\mathrm{Ge}\,\mathrm{Ko}_d}{\mu_m^2 - \mathrm{Lu}_d\,\mathrm{Ge}}\right) P_{m1} - \varepsilon\,\mathrm{Ko}_m Q_{m1}}{P_{m1} Q_{m2} - P_{m2} P_{m1}},$$

$$A_m = \frac{2 \sin \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}}}{\frac{\mu_m}{\sqrt{\mathrm{Lu}_d}} + \sin \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}} \cos \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}}} .$$

The characteristic roots μ_n are determined by the equation

$$P_{n1} Q_{n2} - P_{n2} Q_{n1} = 0$$

and the characteristic roots μ_m from the equation

$$\cot \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}} = \frac{1}{\mathrm{Bi}_d} \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}} .$$

The numerical values of the characteristic roots and certain coefficients can be found by using the corresponding tables and graphs of this chapter.

(b) *Solutions for an infinite plate* ($\mathrm{Pn}_d = 0$).

Since the quantity of the gaseous products of decomposition in the material is small in comparison with the content of associated matter, and the fraction of the total flux of transferred matter represented by the thermo-diffusion of gaseous products is inconsiderable, it is of interest to give a solution of the above problem for $\mathrm{Pn}_d = 0$. In this case the solutions (7.8)–(7.10) remain valid, and the solution (7.11) assumes the following form:

$$W_2(X, \mathrm{Fo}) = -\left\{1 + \right.$$

$$+ \frac{1}{W_0}\left(1 - \frac{\cos \sqrt{(\mathrm{Ge})}\, X}{\cos \sqrt{\mathrm{Ge}} - \frac{1}{\mathrm{Bi}_d} \sqrt{\mathrm{Ge}} \sin \sqrt{\mathrm{Ge}}}\right) \exp\left(-\mathrm{Lu}_d \mathrm{Ge}\, \mathrm{Fo}\right) -$$

$$- \frac{1}{\mathrm{Lu}_d} \sum_{m=1}^{\infty} A_m \left[1 + \frac{1}{W_0} \frac{\mathrm{Lu}_d \mathrm{Ge}}{\mu_m^2(\mu_m^2 - \mathrm{Lu}_d \mathrm{Ge})}\right] \times$$

$$\left. \times \cos \frac{\mu_m}{\sqrt{\mathrm{Lu}_d}} X \exp\left(-\mu_m^2 \mathrm{Fo}\right)\right\} .$$

§ 8. Heat and Mass Exchange in a Layer of a Moist Body

The system of differential equations (I – VI) in its essence describes the transfer of heat and mass in a single body. It is of interest to generalize this system of equations for the case where the process of heat and mass exchange takes place in a layer. Such a system in the first place permits us to avoid a number of limitations which are imposed on the analytical description of processes taking place in layers. In most of the works on heat exchange and drying, the particles of a layer are assumed to be so small, or the heat conductivity and the mass diffusion coefficient in them so great, that the gradients of the transfer potentials of heat and mass in the particles are neglected. Consequently the moist material becomes a continuous medium with temperature and potential of mass continuously changing in one direction. Real processes take place for the most part in a layer whose elements are not small in comparison with the dimensions of the layer. If at the same time the temperature of the heat carrier is high, then the gradients of the transfer potentials in the material are considerable; in this case the model of the layer as a continuous medium becomes incorrect. The generalization of the system (I – VI) must lead to the establishment of a connection between the temperature of the surface of the body and the temperature of the heat carrier. This approach also permits a more complete and accurate picture of the changing temperature of the medium, and hence may be considered as a further and more penetrating treatment of the material of Sections 2 and 3.

(a) *Statement of the problem. System of differential equations.* We have a layer of moist material consisting of particles of spherical form of radius R. A heat carrier is blown through the layer in direction η with constant velocity v. The temperature up to the layer is t_c^0. The initial transfer potentials in the particles of the layer (i.e. up to the contact of the particles with the heat carrier) are t_0 and θ_0. The porosity of the layer is P.

Neglecting the influence of radiation within the layer and contact heat conduction between the particles, the system of transfer equations may be written:

$$\frac{\partial [r\iota(r,\tau)]}{\partial \tau} = a_q \frac{\partial^2 [r\iota(r,\tau)]}{\partial r^2} + \frac{\varepsilon \varrho\, c'_T}{c_q} \frac{\partial [r\,\theta(r,\tau)]}{\partial \tau},$$

$$\frac{\partial [r\,\theta(r,\tau)]}{\partial \tau} = a_m \frac{\partial^2 [r\,\theta(r,\tau)]}{\partial r^2} + a_m \delta \frac{\partial^2 [r\iota(r,\tau)]}{\partial r^2},$$

$$-\lambda_q \frac{\partial \iota(R,\tau)}{\partial r} + \alpha\,[t_c(\eta,\tau) - t(R,\tau)] - (1-\varepsilon)\,\varrho\, q_m = 0, \quad (8.1)$$

$$\lambda_m \left[\frac{\partial \theta(R,\tau)}{\partial r} + \delta \frac{\partial \iota(R,\tau)}{\partial r}\right] + q_m = 0,$$

$$\frac{\partial t_c(\eta,\tau)}{\partial \tau} + v \frac{\partial t_c(\eta,\tau)}{\partial \eta} + \frac{\alpha F}{c_p \gamma_p p}\,[t_c(\eta,\tau) - t(R,\tau)] = 0,$$

$$\frac{\partial \iota(0,\tau)}{\partial r} = \frac{\partial \theta(0,\tau)}{\partial r} = 0$$

for

$$\eta = 0 \quad t_c = t_c^0 = \text{const},$$

$$\eta = v\tau \;\; t = t_0 = \text{const}, \;\; \theta = \theta_0 = \text{const}.$$

(b) *Criterial equations for heat and mass exchange in the layer*. From the system of equations (8.1) we can obtain new criteria of similarity:

a criterion of dimensionless time, characterizing the transformation of the time scale,

$$\text{Ho} = \frac{v\tau}{R};$$

the Margoulis criterion, measuring the ratio of the specific heat communicated to the material to the heat carried through the layer by the heat carrier,

$$\text{Mr} = \frac{a}{v\, c_p\, \gamma_p};$$

Miniovich's criterion, composed of magnitudes defining the geometry of the layer,

$$\text{Mn} = \frac{FR}{p}.$$

Since $\mathrm{Ho} = \mathrm{FoPe}$, where $\mathrm{Pe} = vR/a_q$ is Peclet's criterion, the criterial equations for a layer of moist matter can be written:

$$T = T(\mathrm{Fo}, \mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Bi}_m, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Pe}, \mathrm{Mr}, \mathrm{Mn}, W, V, x/R, \eta/R) \tag{8.2}$$

$$\theta = \theta(\mathrm{Fo}, \mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Bi}_m, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Pe}, \mathrm{Mr}, \mathrm{Mn}, W, V, x/R, \eta/R) . \tag{8.3}$$

The number of independent criteria in equations (8.2) and (8.3) is considerably smaller, since there are connections between the individual criteria.

(c) *Solutions for the temperature fields in the material and the heat carrier at* $\varrho = 0$

$$T = \frac{t(r,\tau) - t_0}{t_c^0 - t_0} = 1 - \sum_{n=1}^{\infty} A_n \left(1 - \mathrm{Mr}\,\mathrm{Mn}\,\frac{\eta}{R}\right) \times$$

$$\times \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left[-\mu_n^2 K_3\left(\mathrm{Pe}\,\mathrm{Fo} - \frac{\eta}{R}\right)\right] -$$

$$- \sum_{n=1}^{\infty} A_n^2 \frac{\sin \mu_n}{\mu_n} \left\{\left[\frac{\mu_n^2}{\mu_n^2 + \mathrm{Bi}_q(\mathrm{Bi}_q - 1)} + \right.\right.$$

$$\left. + \mu_n^2 K_3\left(\mathrm{Pe}\,\mathrm{Fo} - \frac{\eta}{R}\right) + \frac{\mathrm{Bi}_q}{2}\right] \times$$

$$\left. \times \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} - \cos \mu_n \frac{r}{R}\right\} \mathrm{Mr}\,\mathrm{Mn}\,\frac{\eta}{R} \times$$

$$\times \exp\left[-\mu_n^2 K_3\left(\mathrm{Pe}\,\mathrm{Fo} - \frac{\eta}{R}\right)\right] + \ldots$$

$$T_c = \frac{t_c(\eta,\tau) - t_0}{t_c^0 - t_0} = 1 - \sum_{n=1}^{\infty} A_n\,\mathrm{Mr}\,\mathrm{Mn}\,\frac{\eta}{R} \times$$

$$\times \exp\left[-\mu_n^2 K_3\left(\mathrm{Pe}\,\mathrm{Fo} - \frac{\eta}{R}\right)\right] + \ldots$$

Here

$$A_n = \frac{2\,(\sin\mu_n - \mu_n\cos\mu_n)}{\mu_n - \sin\mu_n\cos\mu_n} = \frac{2\,\mathrm{Bi}_q \sqrt{[\mu_n^2 + (\mathrm{Bi}_q - 1)^2]}}{\mu_n^2 + \mathrm{Bi}_q(\mathrm{Bi}_q - 1)},$$

$$K_3 = \frac{\mathrm{Mr}}{\mathrm{Bi}_q}\,\frac{c_p\,\gamma_p}{c_q\,\gamma_0}.$$

μ_n are the roots of the characteristic equation

$$\tan\mu = -\frac{\mu}{\mathrm{Bi}_q - 1}.$$

CHAPTER 3

Non-stationary Fields of Transfer Potential of Heat and Mass with Boundary Conditions of the First Kind

In certain processes, in determining the coefficients of heat and mass transfer in thermal and chemical apparatus, it is necessary to maintain constant values for the potentials of transfer of heat and mass on the surface of the material. The mathematical analogue of these conditions consists of boundary conditions of the first kind. Solutions of the system of differential equations of heat and mass transfer (I), (II) in this case can be obtained from the solutions of the corresponding problems of the preceding chapter by putting the criteria Bi_q and Bi_m equal to infinity. The solutions of (I), (II) under the condition of the constancy of the integral transfer potential of mass

$$\frac{\Gamma+1}{R^{\Gamma+1}}\int_0^R \xi^{\Gamma}\,\theta(\xi,\tau)\,d\xi = \theta_0 = \text{const} \qquad \text{(VII)}$$

and of the constancy of the temperature on the surface of the body

$$t(R,\tau) = 0, \qquad \text{(VIII)}$$

are given below. The results of § 1 were obtained on the basis of Kuznetsov's solutions, solutions of § 2(d) are given by Sizyakova, and § 2(a) and (c) by Kuznetsov.

§ 1. Non-stationary Fields of Transfer Potential of Heat and Mass with Constant Mass Content of the Body

(*a*) *Solution for an infinite plate* ($\xi = x$, $\Gamma = 0$, $t(x, 0) = t_0 = \text{const}$, $\theta(x, 0) = \theta_0 = \text{const}$). The general transform solution

of (I), (II) can be written thus (see Chapter 2, § 1):

$$t_L(x, s) = \frac{t_0}{s} + B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x + B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x, \quad (1.1)$$

$$\theta_L(x, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon \varrho} B_1 (1 - \nu_1^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 x +$$

$$+ \frac{c}{\varepsilon \varrho} B_2 (1 - \nu_2^2) \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 x, \quad (1.2)$$

where

$$\nu_j^2 = \frac{1}{2}\left\{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + (-1)^j \times \right.$$

$$\left. \times \sqrt{\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}\right]}\right\},$$

and B_1 and B_2 are constants with respect to x. The latter are obtained from the boundary conditions (VII) and (VIII) which, in transform, will have the form

$$\frac{1}{R}\int_0^R \theta(x, s)\, dx = \frac{\theta^0}{s}, \quad (1.3)$$

$$t_L(R, s) = 0. \quad (1.4)$$

Adjusting solutions (1.1), (1.2) to the conditions (1.3), (1.4), we obtain

$$\nu_2(1 - \nu_1^2) B_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R + (1 - \nu_2^2) \nu_1 B_2 \times$$

$$\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 R = 0, \quad (1.5)$$

$$B_1 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R + B_2 \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 R + \frac{t_0}{s} = 0. \quad (1.6)$$

From equality (1.6) we determine B_1 and substitute in (1.5); we obtain

$$B_2 = \nu_2(1 - \nu_1^2)\, t_0 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R \times$$

$$\times s\left[\nu_1(1 - \nu_2^2) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R -\right.$$

$$- \nu_2(1-\nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\Big]^{-1},$$

then

$$B_1 = \nu_1(1-\nu_2^2)\,t_0\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\times$$

$$\times s\Big[\nu_1(1-\nu_2^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R -$$

$$- \nu_2(1-\nu_1^2)\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\Big]^{-1}$$

Hence, the transforms of the functions of the distribution of transfer potentials take the form;

$$t_L(x,s) - \frac{t_0}{s} = -t_0\,\frac{K_4\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 x -}{s\Big[K_4\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R -}$$

$$\frac{-\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 x}{-\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\Big]} = -t_0\frac{\Phi_1(s)}{\Psi(s)}, \qquad (1.7)$$

$$\frac{\Theta_0}{s} - \Theta_L(x,s) = \frac{c}{\varepsilon Q}(1-\nu_2^2)\,t_0\,\frac{\frac{\nu_1}{\nu_2}\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\times}{s\Big[K_4\sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\times}$$

$$\frac{\times\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 x - \sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 x}{\times\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R - \sinh\sqrt{\left(\frac{s}{a_q}\right)}\nu_1 R\cosh\sqrt{\left(\frac{s}{a_q}\right)}\nu_2 R\Big|}$$

$$= \frac{ct_0}{\varepsilon Q}(1-\nu_2^2)\frac{\Phi_2(s)}{\Psi(s)}, \qquad (1.8)$$

where

$$K_4 = \frac{\nu_1}{\nu_2}\left(\frac{1-\nu_2^2}{1-\nu_1^2}\right).$$

To determine the inversions $t(x,\tau)$ and $\theta(x,\tau)$, we make use of the expansion theorem, since the expressions (1.7) and

(1.8) can be written in the form of the ratios of generalized polynomials in s, where the degree of the polynomials in the numerator is one less than that of the polynomial in the denominator.

Let us find the roots of $\psi(s)$, for which purpose we equate it to 0. We have:

(1) zero root $s = 0$,

(2) roots which satisfy the equation

$$K_4 \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} \times$$

$$\times \nu_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_2 R = 0 .$$

The latter equation can be written in the form

$$K_4 \sin N\mu \cos \mu - \sin \mu \cos N\mu = 0 , \qquad (1.9)$$

where

$$\mu = i \sqrt{\left(\frac{s}{a_q}\right)} \nu_1 R, \; N = \frac{\nu_2}{\nu_1} .$$

Equation (1.9) has an infinite number of real simple roots which are the general roots of the system of equations (1.10):

$$\left.\begin{aligned} \sin \mu &= 0 \\ \sin N\mu &= 0 \end{aligned}\right\}, \qquad (1.10)$$

and also the roots of the equation

$$\cot N\mu - K_4 \cot \mu = 0 . \qquad (1.11)$$

The system of equations (1.10) will have general roots only in the case where N is a rational fraction. The values of the roots of the characteristic equation (1.11) can be obtained by a graphic-analytical method. Hence, for equation (1.7) the expansion theorem can be written in the following form:

$$t(x, \tau) = -t_0 \left[\lim_{s \to 0} \frac{\Phi_1(s)}{\psi'(s)} + \sum_{l=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} e^{s_l \tau} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} e^{s_n \tau} \right] . \qquad (1.12)$$

In order to determine the first term of (1.12), we write the ratio $\Phi_1(s)/\psi(s)$ in the form

$$\frac{\Phi_1(s)}{\psi(s)} = \frac{K_4\left\{1+\frac{1}{2!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_1 x\right]^2+\dots\right\}-}{s\left\{K_4\left\{1+\frac{1}{2!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_1 R\right]^2+\dots\right\}-\right.}$$

$$-\left\{1+\frac{1}{2!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_2 x\right]^2+\dots\right\}\frac{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_1 R}{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_2 R}$$

$$\left.-\left\{1+\frac{1}{2!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_2 R\right]^2+\dots\right\}\frac{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_1 R}{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_2 R}\right\}, \qquad (1.13)$$

where

$$\frac{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_1 R}{\sinh\sqrt{\left(\frac{s}{a_q}\right)}r_2 R} = \frac{1+\frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_1 R\right]^2+\dots}{1+\frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)}r_2 R\right]^2+\dots}\frac{r_1}{r_2}.$$

Differentiating the denominator of (1.13) with respect to s and passing to the limit for $s \to 0$, we obtain

$$\lim_{s\to 0}\frac{\Phi_1(s)}{\psi'(s)} = 1\,.$$

In the second term of (1.12) the summation is carried out for the general roots of the system (1.10). Let us denote the general roots by $\mu_l = r_1 l\pi$, for $l = 1, 2, 3, \dots$. Then, passing from hyperbolic functions to trigonometrical functions, we have

$$\Phi_1(s_l) = -i\left(K_4 \sin N\mu_l \cos\mu_l\frac{x}{R} - \sin\mu_l \cos N\mu_l\frac{x}{R}\right)$$

$$= -i\left(K_4 \sin r_2 l\pi \cos r_1 l\pi\frac{x}{R} - \sin r_1 l\pi \cos r_2 l\pi\frac{x}{R}\right) = 0\,,$$

and

$$\psi'(s_l) = -i\left(K_4 \frac{N\mu_l}{2}\cos N\mu_l \cos\mu_l - \frac{\mu_l}{2}\cos\mu_l \cos N\mu_l\right)$$

$$= -i\frac{\mu_l}{2}(K_4 N - 1)\cos N\mu_l \cos\mu_l \neq 0\,.$$

Thus,

$$\sum_{n=l}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} \exp(s_l \tau) = 0\,.$$

In the third term the summation is by the positive roots of equation (1.11) and consequently,

$$\sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} e^{s_n \tau}$$

$$= 2\sum_{n=1}^{\infty} \frac{K_4 \sin N\mu_n \cos\mu_n \frac{x}{R} - \sin\mu_n \cos N\mu_n \frac{x}{R}}{\mu_n[(NK_4 - 1)\cos\mu_n \cos N\mu_n + (N - K_4)\sin\mu_n \sin N\mu_n]} \times$$

$$\times \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right).$$

We substitute the values found for the terms of equality (1.12), when for the temperature distribution we obtain

$$\frac{t(x,\tau)}{t_0}$$

$$= -\sum_{n=1}^{\infty}\left(C_{1n}\cos\mu_n \frac{x}{R} - C_{2n}\cos N\mu_n \frac{x}{R}\right)\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right), \quad (1.14)$$

where

$$C_{1n} = \frac{2K_4 \sin N\mu_n}{\mu_n[(NK_4 - 1)\cos\mu_n \cos N\mu_n + (N - K_4)\sin\mu_n \sin N\mu_n]},$$

$$C_{2n} = \frac{2\sin\mu_n}{\mu_n[(NK_4 - 1)\cos\mu_n \cos N\mu_n + (N - K_4)\sin\mu_n \sin N\mu_n]},$$

μ_n are the roots of the characteristic equation (1.11).

We will now determine the distribution of the transfer potential of mass in an infinite plate. Since the roots of the denominator of (1.8) remain as before, the expansion theorem

can be written thus:

$$\theta_0 - \theta(x,\tau) = \frac{\sigma}{c\varrho} I_0(1-\nu_2^2)\left|\lim_{s\to 0}\frac{\varphi_2(s)}{\psi'(s)} + \right.$$

$$\left. + \sum_{l=1}^{\infty}{}' \frac{\varphi_2(s_l)}{\psi'(s_l)} e^{s_l\tau} + \sum_{n=1}^{\infty}\frac{\varphi_2(s_n)}{\psi'(s_n)} e^{s_n\tau}\right|. \qquad (1.15)$$

Making use of a method analogous to that which we applied in calculating the two first terms of (1.12), we can show that the corresponding terms of (1.15) become zero. The third term gives

$$\sum_{n=1}^{\infty}{}' \frac{\varphi_2(s_n)}{\psi'(s_n)} e^{s_n\tau}$$

$$= 2\sum_{n=1}^{\infty}\frac{\left[\frac{1}{N}\sin N\mu_n\cos\mu_n\frac{x}{R} - \sin\mu_n\cos N\mu_n\frac{x}{R}\right]}{\mu_n[(NK_4-1)\cos\mu_n\cos N\mu_n + (N-K_4)\sin\mu_n\sin N\mu_n]}\times$$

$$\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right).$$

Thus, for the transfer potential field of mass we finally obtain

$$\frac{\theta_0-\theta(x,\tau)}{\theta_0}$$

$$= \frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty}{}'\left(C_{1n}^*\cos\mu_n\frac{x}{R} - C_{2n}^*\cos N\mu_n\frac{x}{R}\right)\times$$

$$\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right), \qquad (1.16)$$

where

$$C_{1n}^* = \frac{1}{NK_4}C_{1n},\ C_{2n}^* = C_{2n},$$

$\mathrm{Ko}_1 = \varrho c_1' \theta_0/c_q t_0$ is the Kossovich criterion.

μ_n are the roots of the characteristic equation (1.11).

The mean values of the transfer potentials are determined by the expressions:

$$\bar{t}(\tau) = \frac{1}{R}\int_0^R t(x,\tau)\,dx,\ \bar{\theta}(\tau) = \frac{1}{R}\int_0^R \theta(x,\tau)\,dx.$$

For the mean non-dimensional temperature of the body we have

$$\frac{\bar{t}(\tau)}{t_0} = -\sum_{n=1}^{\infty} D_n \exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right) \tag{1.17}$$

and for the mean non-dimensional potential of transfer of mass

$$\frac{\theta_0 - \bar{\theta}(\tau)}{\theta_0} = \frac{1 - r_2^2}{\varepsilon\,\mathrm{Ko}_1} \sum_{n=1}^{\infty} D_n^* \exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right), \tag{1.18}$$

where

$$D_n = C_{1n}\frac{\sin\mu_n}{\mu_n} - C_{2n}\frac{\sin N\mu_n}{N\mu_n},$$

$$D_n^* = C_{1n}^*\frac{\sin\mu_n}{\mu_n} - C_{2n}^*\frac{\sin N\mu_n}{N\mu_n}.$$

The local and mean non-dimensional rates of change of the transfer potentials are obtained by differentiating equations (1.14), (1.16)–(1.18) with respect to the non-dimensional time i.e.

$$\frac{d\left[\frac{t(x,\tau)}{t_0}\right]}{d\,\mathrm{Fo}}$$

$$= \sum_{n=1}^{\infty} \frac{\mu_n^2}{r_2^1}\left(C_{1n}\cos\mu_n\frac{x}{R} - C_{2n}\cos N\mu_n\frac{x}{R}\right)\exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\bar{t}(\tau)}{t_0}\right]}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty} \frac{\mu_n^2}{r_1^2} D_n \exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\theta(x,\tau)}{\theta_0}\right]}{d\,\mathrm{Fo}}$$

$$= \frac{1 - r_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty} \frac{\mu_n^2}{r_1^2}\left(C_{1n}^*\cos\mu_n\frac{x}{R} - C_{2n}^*\cos N\mu_n\frac{x}{R}\right)\exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\bar{\theta}(\tau)}{\theta_0}\right]}{d\,\mathrm{Fo}} = \frac{1 - r_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty} \frac{\mu_n^2}{r_1^2} D_n^* \exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right).$$

(b) *Solution for the sphere* ($\xi = r$, $\Gamma = 2$, $t(r, 0) = t_0 = \text{const}$, $\theta(r, 0) = \theta_0 = \text{const}$).

The general solution of (I), (II) in transform can be represented in the form (see Chapter 2 § 1 (a)):

$$t_L(r, s) = \frac{t_0}{s} + \frac{B_1}{r} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 r + \\ + \frac{B_2}{r} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 r , \qquad (1.19)$$

$$\theta_L(r, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon \varrho} (1 - r_1^2) \frac{B_1}{r} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 r + \\ + \frac{c}{\varepsilon \varrho} (1 - r_2^2) \frac{B_2}{r} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 r . \qquad (1.20)$$

Adjusting equations (1.19) and (1.20) to the boundary conditions (VII) and (VIII) in transform, we obtain:

$$B_1 \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R + B_2 \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R + \frac{R t_0}{s} = 0 , \quad (1.21)$$

$$(1 - r_1^2) B_1 \frac{\sqrt{\left(\frac{s}{a_q}\right)} r_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R}{\left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right]^2} + \\ + (1 - r_2^2) B_2 \frac{\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R}{\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right]^2} = 0 . \qquad (1.22)$$

Solving simultaneously (1.21) and (1.22) for B_1 and B_2 we obtain

$$B_1 = -\frac{K_4}{N} R t_0 \left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right] \times \\ \times s \left\{\frac{K_4}{N} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R \times \right. \\ \left. \times \left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right] -\right.$$

$$- \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R \left[\sqrt{\left(\frac{s}{a_q}\right)} v_1 R \times\right.$$

$$\left.\times \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R\right]\Big\}^{-1},$$

$$B_2 = Rt_0 \left[\sqrt{\left(\frac{s}{a_q}\right)} v_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R\right] \times$$

$$\times s \left\{\frac{K_4}{N} \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R \times\right.$$

$$\times \left[\sqrt{\left(\frac{s}{a_q}\right)} v_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R\right] -$$

$$- \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R \left[\sqrt{\left(\frac{s}{a_q}\right)} v_1 R \times\right.$$

$$\left.\left.\times \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R\right]\right\}^{-1}.$$

Hence, the transforms of the functions of the distribution of transfer potentials of heat and mass can be written, after a few simple transformations, in the form:

$$t_L(r, s) - \frac{t_0}{s} = - \frac{Rt_0}{r} \times$$

$$\times \frac{\frac{K_4}{N} \left[\sqrt{\left(\frac{s}{a_q}\right)} v_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R\right] \times}{s \left\{\frac{K_4}{N} \left[\sqrt{\left(\frac{s}{a_q}\right)} v_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R\right] \times\right.}$$

$$\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 r - \left[\sqrt{\left(\frac{s}{a_q}\right)} v_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R -\right.$$

$$\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R - \left[\sqrt{\left(\frac{s}{a_q}\right)} v_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R -\right.$$

$$\left.\left.- \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 r\right.$$

$$\left.- \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_1 R\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} v_2 R\Big\} = - \frac{Rt_0}{r} \frac{\Phi_1(s)}{\psi(s)}, \quad (1.23)$$

$$\frac{\theta_0}{s} - \theta_L(r, s) = \frac{c}{\varepsilon \varrho} \frac{Rt_0}{r} \frac{1 - v_2^2}{N^2} \times$$

$$\times \frac{\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right] \times}{s\left\{\frac{K_4}{N}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right] \times\right.}$$

$$\frac{\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 r - N^2 \left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R -\right.}{\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R - \left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R -\right.}$$

$$\frac{-\left.\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 r}{-\left.\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\Big|}$$

$$= \frac{c}{\varepsilon_0} \frac{R l_0}{r} \frac{(1 - r_1^2)}{N^2} \frac{\Phi_2(s)}{\psi(s)}, \qquad (1.24)$$

where the values of $\Phi_1(s)$, $\Phi_2(s)$ and $\psi(s)$ follow immediately from consideration of the expressions (1.23) and (1.24); K_4 and N are determined above.

In order to pass from the operational solutions (1.23) and (1.24) to the inversions $t(r, \tau)$ and $\theta(r, \tau)$ we make use of the expansion theorem. The ratios $\Phi_1(s)/\psi(s)$ and $\Phi_2(s)/\psi(s)$ can be regarded as ratios of generalized polynomials in s, to which the expansion theorem is applicable. The common denominator $\psi(s)$ has the roots:

(a) zero root ($s = 0$),

(b) the roots of the equation

$$\frac{K_4}{N}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right] \times$$

$$\times \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R - \left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R \cosh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R -\right.$$

$$-\left.\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right] \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_2 R = 0,$$

which can also be written in the form

$$\frac{K_4}{N}(\sin N\mu - N\mu \cos N\mu) \sin \mu - (\sin \mu - \mu \cos \mu) \sin N\mu = 0,$$

where

$$\mu = i \sqrt{\left(\frac{s}{a_q}\right)} v_1 R .$$

The last equation is satisfied by the general roots of the system of equations

$$\left.\begin{aligned} \sin \mu &= 0 \\ \sin N\mu &= 0 \end{aligned}\right\}, \tag{1.25}$$

if this system has general roots (i.e. if N is a rational fraction), and by the roots of the equation

$$K_4 \mu \cot N\mu - \mu \cot \mu - \frac{K_4}{N} + 1 = 0 . \tag{1.26}$$

Applying the expansion theorem to equation (1.23), we obtain

$$t(r,\tau) - t_0 = -\frac{Rt_0}{t}\left[\lim_{s\to 0}\frac{\Phi_1(s)}{\psi'(s)} + \right.$$

$$\left. + \sum_{l=1}^{\infty}\frac{\Phi_1(s_l)}{\psi'(s_l)} e^{s_l\tau} + \sum_{n=1}^{\infty}\frac{\Phi_1(s_n)}{\psi'(s_n)} e^{s_n\tau}\right]. \tag{1.27}$$

Here

s_l are the general roots of the system of equations (1.25),

s_n are the roots of the equation (1.26).

If in the ratio $\Phi_1(s)/\psi(s)$ of equation (1.27) we substitute $s = 0$, we obtain an indeterminate result. Hence we first expand the hyperbolic functions in $\Phi_1(s)$ and $\psi(s)$ as infinite series, and then carry out the reduction, obtaining as a result.

$$\frac{\Phi_1(s)}{\psi(s)} = \frac{\frac{K_4}{N}\left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q} v_2^2 R^2 \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right] \times}{s\left[\frac{K_4}{N}\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q} v_2^2 R^2 \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right] \times}$$

$$\frac{\times v_2^2 - \left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q} v_1^2 R^2 \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right] \times}{\times v_2^2 - \left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q} v_1^2 R^2 \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right] \times}$$

$$\times r_1^2 \frac{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2\right]^2 r^2 + \dots}{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_1\right]^2 r^2 + \dots} \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 r}{\times r_1^2 \frac{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_2 R\right]^2 + \dots}{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right]^2 + \dots} \sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R},$$

where the last factor is

$$\frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 r}{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_1 R} = \frac{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 r\right]^2 + \dots}{1 + \frac{1}{3!}\left[\sqrt{\left(\frac{s}{a_q}\right)} r_1 R\right]^2 + \dots} \frac{r}{R}.$$

Differentiating the denominator of the ratio obtained with respect to s, and passing to the limit of the ratio $\Phi_1(s)/\psi'(x)$ for $s \to 0$, we obtain for the first term of (1.27)

$$\lim_{s \to 0} \frac{\Phi_1(s)}{\psi'(s)} = \frac{r}{R}.$$

Let us find the second term of (1.27). The general roots of the system of equations (1.25) have the form $\mu_l = \beta l \pi$ for $l = 1, 2, 3, \dots$, if $N = r_2/r_1 = \alpha/\beta$ is a rational fraction (α and β are integers). It is easy to show by a series of simple transformations that

$$\Phi_1(s_l) = \mu_l \cos \mu_l \sin N \mu_l \frac{r}{R} - \frac{K_4}{N} N \mu_l \cos N \mu_l \sin \mu_l \frac{r}{R},$$

$$\psi'(s_l) = \frac{1}{2} \mu_l N \mu_l \cos \mu_l \cos N \mu_l \left(1 - \frac{K_4}{N}\right).$$

Since

$$\cos \mu_l = \cos \beta l \pi = (-1)^{\beta l} \text{ and } \cos N \mu_l = \cos \alpha l \pi = (-1)^{\alpha l},$$

for the second term of equality (1.27) we finally obtain

$$\sum_{l=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} \exp(s_l \tau) =$$

$$= 2 \sum_{l=1}^{\infty} \frac{(-1)^{\beta l} \beta \sin \alpha l \pi \frac{r}{R} - \frac{K_4}{N} (-1)^{\alpha l} \alpha \sin \beta l \pi \frac{r}{R}}{\alpha \beta l \pi (-1)^{\beta l} (-1)^{\alpha l} \left(1 - \frac{K_4}{N}\right)} \times$$

$$\times \exp\left(- \frac{\beta^2 l^2 \pi^2 a_q \tau}{r_1^2 R^2}\right).$$

For the third term of (1.27) calculation gives

$$\Phi_1(s_n) = \frac{K_4}{N} (\sin N\mu_n - N\mu_n \cos N\mu_n) \sin \mu_n \frac{r}{R} -$$

$$- (\sin \mu_n - \mu_n \cos \mu_n) \sin N\mu_n \frac{r}{R},$$

$$\varphi'(s_n) = \frac{1}{2} \left[\mu_n \frac{K_4}{N} \cos \mu_n (\sin N\mu_n - N\mu_n \cos N\mu_n) - \right.$$

$$- N\mu_n \cos N\mu_n (\sin \mu_n - \mu_n \cos \mu_n) +$$

$$\left. + (NK_4 - 1) \mu_n^2 \sin \mu_n \sin N\mu_n \right].$$

Substituting for the respective terms in equality (1.27) the corresponding values obtained above we have, for the distribution of transfer potentials of heat in a sphere,

$$\frac{t(r, \tau)}{t_0} = - \left[\sum_{l=1}^{\infty} \left(E_{1l} \frac{\sin \alpha \pi l \frac{r}{R}}{\frac{r}{R}} - E_{2l} \frac{\sin \beta \pi l \frac{r}{R}}{\frac{r}{R}} \right) \times \right.$$

$$\times \exp\left(- \frac{\beta^2 \pi^2 l^2}{r_1^2} \mathrm{Fo}\right) +$$

$$+ \sum_{n=1}^{\infty} \left(C_{1n} \frac{\sin \mu_n \frac{r}{R}}{\frac{r}{R}} - C_{2n} \frac{\sin N\mu \frac{r}{R}}{\frac{r}{R}} \right) \times$$

$$\left. \times \exp\left(- \frac{\mu_n^2}{r_1^2} \mathrm{Fo}\right) \right], \qquad (1.28)$$

where

$$E_{1l} = \frac{2N}{(-1)^{\alpha l} (N - K_4) \alpha \pi l}, \quad E_{2l} = \frac{2K_4}{(-1)^{\beta l} (N - K_4) \beta \pi l},$$

$$C_{1n} = \frac{2\frac{K_4}{N}\varphi(N\mu_n)}{\psi_n}, \quad C_{2n} = \frac{2\varphi(\mu_n)}{\psi_n},$$

$$\varphi(N\mu_n) = \sin N\mu_n - N\mu_n \cos N\mu_n,$$

$$\varphi(\mu_n) = \sin \mu_n - \mu_n \cos \mu_n,$$

$$\psi_n = \frac{K_4}{N}\mu_n \cos \mu_n \varphi(N\mu_n) - N\mu_n \cos N\mu_n \varphi(\mu_n) +$$

$$+ (NK_4 - 1)\mu_n^2 \sin \mu_n \sin N\mu_n.$$

The distribution function of the transfer potential of mass in a sphere can be obtained by a method analogous to that used in obtaining the function of temperature distribution (1.28). The solution gives

$$\frac{\theta_0 - \theta(r,\tau)}{\theta_0} = \frac{1-r_2^2}{\varepsilon \mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}\left(E_{11}^* \frac{\sin \alpha\pi l \frac{r}{R}}{\frac{r}{R}} - E_{21}^* \frac{\sin \beta\pi l \frac{r}{R}}{\frac{r}{R}}\right) \times\right.$$

$$\times \exp\left(-\frac{\beta^2\pi^2 l^2}{r_1^2}\mathrm{Fo}\right) +$$

$$\left. + \sum_{n=1}^{\infty}\left(C_{11}^* \frac{\sin \mu_n \frac{r}{R}}{\frac{r}{R}} - C_{2n}^* \frac{\sin N\mu_n \frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\frac{\mu_n^2}{r_1^2}\mathrm{Fo}\right)\right], \quad (1.29)$$

where

$$E_{11}^* = E_{11}, \; E_{21}^* = \frac{1}{NK_4}E_{21},$$

$$C_{1n}^* = \frac{1}{NK_4}C_{1n}, \; C_{2n}^* = C_{2n},$$

In the first term summation is by the general roots of equation (1.25), and in the second by the roots of the characteristic equation (1.26). α and β are integers whilst $N = r_2/r_1 = \alpha/\beta$ is a rational fraction.

The mean values of the transfer potentials for a body in the form of a sphere are determined from the expressions

$$\tilde{t}(\tau) = \frac{3}{R^3}\int_0^R r^2 t(r,\tau)\,dr\,, \quad \overline{\theta}(\tau) = \frac{3}{R^3}\int_0^R r^2\theta(r,\tau)\,dr\,.$$

For non-dimensional values of the potentials we have:

$$\frac{t(\tau)}{t_0} = -\left[\sum_{l=1}^{\infty} F_l \exp\left[-\left(\frac{\beta\pi\, l}{\nu_1}\right)^2 \mathrm{Fo}\right] + \right.$$

$$\left. + \sum_{n=1}^{\infty} D_n \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right], \qquad (1.30)$$

$$\frac{\theta_0 - \theta(\tau)}{\theta_0} = \frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty} F_l^* \exp\left[-\left(\frac{\beta\pi\, l}{\nu_1}\right)^2 \mathrm{Fo}\right] + \right.$$

$$\left. + \sum_{n=1}^{\infty} D_n^* \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right], \qquad (1.31)$$

where

$$F_l = \frac{3\,(\sin\alpha\pi\, l - \alpha\pi\, l\cos\alpha\pi\, l)}{(\alpha\pi\, l)^2}\,E_{1l} - \frac{3\,(\sin\beta\pi\, l - \beta\pi\, l\cos\beta\pi\, l)}{(\beta\pi l)^2}\,E_{2l}\,,$$

$$F_l^* = \frac{3\,(\sin\alpha\pi\, l - \alpha\pi\, l\cos\alpha\pi\, l)}{(\alpha\pi\, l)^2}\,E_{1l}^* - \frac{3\,(\sin\beta\pi\, l - \beta\pi\, l\cos\beta\pi\, l)}{(\beta\pi\, l)^2}\,E_{2l}^*\,,$$

$$D_n = \frac{3\,(\sin\mu_n - \mu_n\cos\mu_n)}{\mu_n^2}\,C_{1n} - \frac{3\,(\sin N\mu_n - N\mu_n\cos N\mu_n)}{(N\mu_n)^2}\,C_{2n}\,,$$

$$D_n^* = \frac{3\,(\sin\mu - \mu_n\cos\mu_n)}{\mu_n^2}\,C_{1n}^* - \frac{3\,(\sin N\mu_n - N\mu_n\cos N\mu_n)}{(N\mu_n)^2}\,C_{2n}^*.$$

The non-dimensional rate of change of the transfer potentials (local and mean) is obtained by the differentiation of (1.28)—(1.31) with respect to the non-dimensional time (Fo):

$$\frac{d}{d\,\mathrm{Fo}}\left[\frac{t(r,\tau)}{t_0}\right] = \sum_{l=1}^{\infty}\left(\frac{\beta\pi\, l}{\nu_1}\right)^2\left(E_{1l}\frac{\sin\alpha\pi\, l\,\dfrac{r}{R}}{\dfrac{r}{R}} - E_{2l}\frac{\sin\beta\pi\, l\,\dfrac{r}{R}}{\dfrac{r}{R}}\right)\times$$

$$\times \exp\left[-\left(\frac{\beta\pi\, l}{\nu_1}\right)^2\mathrm{Fo}\right] +$$

$$+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}\left(C_{1n}\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{2n}\times\frac{\sin N\mu\,\frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d}{d\,\mathrm{Fo}}\left[\frac{\bar{t}(\tau)}{t_0}\right]=\sum_{l=1}^{\infty}\left(\frac{\beta\pi\,l}{\nu_1}\right)F_l\exp\left[-\left(\frac{\beta\pi\,l}{\nu_1}\right)^2\mathrm{Fo}\right]+$$

$$+\sum_{n=1}^{\infty}\frac{\mu_n}{\nu_1^2}D_n\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d}{d\,\mathrm{Fo}}\left[\frac{\theta(r,\tau)}{\theta_0}\right]=\frac{1-\nu_1^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}\left(\frac{\beta\pi\,l}{\nu_1}\right)^2\left(E_{1l}^*\frac{\sin\alpha\pi\,l\,\frac{r}{R}}{\frac{r}{R}}-\right.\right.$$

$$\left.\left.-E_{2l}^*\frac{\sin\beta\pi\,l\,\frac{r}{R}}{\frac{r}{R}}\right)\exp\left[-\left(\frac{\beta\pi\,l}{\nu_1}\right)^2\mathrm{Fo}\right]+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}\times\right.$$

$$\left.\times\left(C_{1n}^*\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{2n}^*\frac{\sin N\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right],$$

$$\frac{d}{d\,\mathrm{Fo}}\left[\frac{\bar{\theta}(\tau)}{\theta_0}\right]^2=\frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}F_1^*\left(\frac{\beta\pi\,l^2}{\nu_1}\right)^2\times\right.$$

$$\left.\times\exp\left[-\left(\frac{\beta\pi\,l}{\nu_1}\right)^2\mathrm{Fo}\right]+\sum_{n=1}^{\infty}D_n^*\frac{\mu_n^2}{\nu_1^2}\exp\left(-\frac{\mu_n^2}{\nu_1^2}\right)\mathrm{Fo}\right].$$

(c) *Analysis of the solutions.* From a consideration of the solutions (1.14), (1.16), (1.28) and (1.29) it follows that the distribution of the transfer potentials in the material is determined by the criteria Fo and Ko_1 (or Ko_2), and also by the non-dimensional quantities ν_1 and ν_2. The latter, in their turn, as was shown in § 1(b) of Chapter 2, are functions of the criteria Lu and $\mathrm{Fe}=\varepsilon\,\mathrm{Ko}_1\,\mathrm{Pn}_1$.

Consequently,

$$\frac{t-t_0}{t_0}=f_1\left(\mathrm{Fo},\mathrm{Lu},\varepsilon\,\mathrm{Ko}_1\,\mathrm{Pn}_1,\frac{x}{R}\right),\qquad(1.32)$$

$$\frac{\theta_0 - \theta}{\theta^0} = f_2\left(\mathrm{Fo}, \mathrm{Ko}_1, \mathrm{Lu}, \varepsilon\mathrm{Ko}_1\mathrm{Pn}_1, \frac{x}{R}\right), \qquad (1.33)$$

i.e. the transfer of heat in the material is inseparable from the transfer of mass and vice versa.

Comparing the expressions (1.32) and (1.33) with the analogous expressions (1.32) and (1.33) of Chapter 2 we are driven to the following conclusion. The transition from the solution of heat and mass transfer problems under boundary conditions of the third kind, to problems corresponding to boundary conditions of the first kind, and to an invariable value of the integral of the transfer potential of mass, simplifies the analytical solution on account of the disappearance of the criteria of surface heat and mass exchange (Bi_q, Bi_m) and the connecting together of the criteria of heat and mass transfer (ε, Ko_1, Pn_1) by means of the Fedorov criterion.

Concluding the present section we will add the calculation of the temperature field and the field of transfer potential of mass (mass-content) for an infinite plate. We take the coefficients of heat and mass transfer as follows:

$c_q = 0{\cdot}45$ kcal/kg. deg, $a_q = 0{\cdot}865 \times 10^{-3}$ m²/hr, $u_0 = 0{\cdot}15$ kg/kg, $a_m = 0{\cdot}13 \times 10^{-3}$ m²/hr, $\delta = 1 \times 10^{-3}$ deg^{-1}, $t_0 = 90$ °C, $\varrho = 560$ kcal/kg, $\varepsilon = 0{\cdot}09$, $t_c = 0$°C.
we obtain

$$\text{the criterion Lu} = a_m/a_q = 0{\cdot}150,$$

$$\text{the criterion Ko}_1 = \varrho u_0/c_q t_0 = 1{\cdot}44,$$

$$\text{the criterion Fe} = \varepsilon\varrho\delta/c_q = 0{\cdot}112,$$

$\nu_1 = 0{\cdot}993$, $\nu_2 = 2{\cdot}61$, $N = 2{\cdot}63$, $K_4 = -147$, $\mu_1 = 1{\cdot}19$, $\mu_2 = 1{\cdot}58$.

The results of the solution of (1.14) and (1.16) for the distribution of the non-dimensional temperature and the non-dimensional transfer potential of mass over the coordinate x/R are collected together in Tables 32 and 33 where we have confined ourselves to two terms of the expansion.

TABLE 32. DISTRIBUTION OF NON-DIMENSIONAL TEMPERATURE IN AN INFINITE PLATE

$\frac{x}{R}$	$\frac{t(x,\tau)}{t_0}$			
	Fo = 0·1	Fo = 0·5	Fo = 1·0	Fo = 1·5
0·0	0·900	0·290	0·096	0·024
0·2	0·918	0·338	0·098	0·029
0·4	0·783	0·287	0·083	0·024
0·6	0·568	0·208	0·060	0·017
0·8	0·295	0·107	0·030	0·009
1·0	0·000	0·000	0·000	0·000

TABLE 33. DISTRIBUTION OF THE NON-DIMENSIONAL MASS TRANSFER POTENTIAL IN AN INFINITE PLATE

$\frac{x}{R}$	$\frac{\theta(x,\tau)-\theta_0}{\theta_0}$			
	Fo = 0·1	Fo = 0·5	Fo = 1·0	Fo = 1·5
0·0	—0·026	0·149	0·079	0·049
0·2	0·002	0·079	0·068	0·041
0·4	0·084	0·060	0·034	0·018
0·6	0·093	0·004	—0·016	—0·013
0·8	—0·041	—0·093	—0·071	—0·042
1·0	—0·275	—0·197	—0·112	—0·059

§ 2. Heat and Mass Transfer with Invariable Mass Content of the Body and in the Absence of Phase Changes

The analytical description of special cases of heat and mass transfer under boundary conditions of the first kind (VII), (VIII) can be obtained from the solutions of the corresponding problems of the previous paragraph by the method of passing to the limit (see § 5 of Chapter 2). The same result follows immediately from the solution of the incomplete system of differential equations (I), (II), ($\delta = 0$, $\varepsilon = 0$, etc.). As an

illustration, let us consider the solution of the incomplete system of equations for the case where there is no phase transformation in the material. This means that the transfer of mass within the material takes place only in the liquid form, i.e. at a temperature of the material of $t \leqslant 50°C$.

(a) *Heat and Mass Transfer in the Sphere* ($\Gamma = 2$, $\xi = r$, $\varepsilon = 0$, $t(r, 0) = t_0 = \text{const.}$, $\theta(r, 0) = \theta_0 = \text{const}$). The initial system of equations (I), (II) for a sphere can be rewritten thus:

$$\frac{\partial [rt(r, \tau)]}{\partial \tau} = a_q \frac{\partial^2 [rt(r, \tau)}{\partial r^2}, \tag{2.1}$$

$$\frac{\partial [r\theta(r, \tau)]}{\partial \tau} = a_m \frac{\partial^2 [r\vartheta(r, \tau)]}{\partial r^2} + a_m \delta \frac{\partial^2 [rt(r, \tau)]}{\partial r^2}. \tag{2.2}$$

The transform for the temperature field (2.1) is

$$t_L(r, s) - \frac{t_0}{s} = -\frac{Rt_0}{r} \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r}{s \sinh \sqrt{\left(\frac{s}{a_q}\right)} R}. \tag{2.3}$$

The inversion of (2.3) is obtained by means of the expansion theorem. If we put

$$\Phi(s) = \sinh \sqrt{\left(\frac{s}{a_q}\right)} r,$$

$$\psi(s) = s \sinh \sqrt{\left(\frac{s}{a_q}\right)} R,$$

then the roots of the polynomial $\psi(s)$ will be:

1. $s = 0$,

2. $s_n = -\frac{\mu_n^2 a_q}{R^2}$, where $\mu_n = i \sqrt{\left(\frac{s_n}{a_q}\right)} R = n\pi (n = 1, 2, 3\ldots)$ —

i.e. the roots of the equation

$$\sinh \sqrt{\left(\frac{s_n}{a_q}\right)} R = 0.$$

Hence the expansion theorem can be written thus:

$$\frac{t(r, \tau)}{t_0} = -\frac{Rt_0}{r} \left[\lim_{s \to 0} \frac{\Phi(s)}{\psi'(s)} + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)} e^{s_n \tau} \right]$$

and the inversion of (2.3) will be

$$\frac{t(r,\tau)}{t_0} = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{2\sin n\pi\frac{r}{R}}{n\pi\frac{r}{R}}\exp\left(-n^2\pi^2\,\mathrm{Fo}\right). \qquad (2.4)$$

For values of the criterion $\mathrm{Fo} > 0{\cdot}1$ the series converges fairly rapidly, and in the calculation we need use only the first term of the expansion. For small values of a non-dimensional time ($\mathrm{Fo} < 0{\cdot}1$) it is necessary to take several terms of the infinite series.

We will now apply the Laplace transformation to equation (2.2). We shall take into account (2.3) and thus obtain

$$a_m u_L''(r,s) - s\left[u_L(r,s) - \frac{r\theta_0}{s}\right] - \frac{Rt_0 a_m \delta \sinh\sqrt{\left(\frac{s}{a_q}\right)}r}{a_q \sinh\sqrt{\left(\frac{s}{a_q}\right)}R} = 0,$$

where

$$u_L(r,s) = r\theta_L(r,s).$$

The solution of this equation gives

$$\theta_L(r,s) = B_1\frac{\cosh\sqrt{\left(\frac{s}{a_m}\right)}r}{r} + B_2\frac{\sinh\sqrt{\left(\frac{s}{a_m}\right)}r}{r} + \frac{\theta_0}{s} + \frac{\delta t_0 R\left[\sinh\sqrt{\left(\frac{s}{a_q}\right)}r - \sqrt{\left(\frac{a_m}{a_q}\right)}\sinh\sqrt{\left(\frac{s}{a_m}\right)}r\right]}{r\left(1-\frac{a_q}{a_m}\right)s\sinh\sqrt{\left(\frac{s}{a_q}\right)}R}. \qquad (2.5)$$

From the transformed condition of symmetry (V), it follows that $B_1 = 0$. The coefficient B_2, which is constant relative to r, can be determined by means of (VII). Adjusting (2.5) to the boundary condition (VII), we obtain

$$B_2 = \frac{\delta t_0 R}{(a_m - a_q)\,s\sinh\sqrt{\left(\frac{s}{a_q}\right)}R} \times$$

$$\times \left\{ \sqrt{\left(\frac{a_m}{a_p}\right)} \left[\sqrt{\left(\frac{s}{a_m}\right)} R \cosh \sqrt{\left(\frac{s}{a_m}\right)} R - \sinh \sqrt{\left(\frac{s}{a_m}\right)} R \right] a_m - \right.$$

$$\left. - \left[\sqrt{\left(\frac{s}{a_q}\right)} R \cosh \sqrt{\left(\frac{s}{a_q}\right)} R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} R \right] a_q \right\} \times$$

$$\times \left[\sqrt{\left(\frac{s}{a_m}\right)} R \cosh \sqrt{\left(\frac{s}{a_m}\right)} R - \sinh \sqrt{\left(\frac{s}{a_m}\right)} R \right]^{-1}.$$

Thus, the solution for the transform of the function of distribution of transfer potential of mass in the sphere has the form:

$$\theta_L(r.s) - \frac{\theta_0}{s} = \frac{\delta t_0 R}{r(a_m - a_q)} \times$$

$$\times \left\{ a_m \sinh \sqrt{\left(\frac{s}{a_q}\right)} r \left(\sqrt{\left(\frac{s}{a_m}\right)} R \cosh \sqrt{\left(\frac{s}{a_m}\right)} R - \sinh \sqrt{\left(\frac{s}{a_m}\right)} R \right] - \right.$$

$$\left. - a_q \left[\sqrt{\left(\frac{s}{a_q}\right)} R \cosh \sqrt{\left(\frac{s}{a_q}\right)} R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} R \right] \sinh \sqrt{\left(\frac{s}{a_m}\right)} r \right] \times$$

$$\times s \sinh \sqrt{\left(\frac{s}{a_q}\right)} R \left[\sqrt{\left(\frac{s}{a_m}\right)} R \cosh \sqrt{\left(\frac{s}{a_m}\right)} R - \sinh \sqrt{\left(\frac{s}{a_m}\right)} R \right]^{-1}$$

$$= \frac{\delta t_0 R}{r(a_m - a_q)} \frac{\Phi(s)}{\psi(s)}. \qquad (2.6)$$

Since the conditions for the expansion theorem are fullfilled in (2.6), we shall make use of this theorem to obtain the original $\theta(r, s)$.

The roots of the polynomial ψs will be:

(1). $s = 0$,

(2). $s_n = -\frac{\mu_n^2 a_q}{R^2}$, where $\mu_n = i \sqrt{\left(\frac{s_n}{a_q}\right)} R = n\pi$ are the roots of the equation $\sinh \sqrt{\left(\frac{s}{a_q}\right)} R = 0$,

(3) $s_l = -\frac{\mu_l^2 a_m}{R^2}$, where $\mu_l = i \sqrt{\left(\frac{s_l}{a_m}\right)} R$ are the roots of the equation

$$\tan \mu_l = \mu_l. \qquad (2.7)$$

Consequently,

$$\theta(r, s) - \theta_0 = \frac{\delta t_0 R}{r(a_m - a_q)} \left[\lim_{s \to 0} \frac{\Phi(s)}{\psi'(s)} + \right.$$

$$+\sum_{n=1}^{\infty}\frac{\varphi(s_n)}{\psi'(s_n)}e^{s_n r}+\sum_{l=1}^{\infty}\frac{\varphi(s_l)}{\psi'(s_l)}e^{s_l r}\Big]. \qquad (2.8)$$

Here

$$\psi'(s)=\sinh\sqrt{\left(\frac{s}{a_q}\right)}R\left[\sqrt{\left(\frac{s}{a_m}\right)}R\cosh\sqrt{\left(\frac{s}{a_m}\right)}R-\right.$$

$$\left.-\sinh\sqrt{\left(\frac{s}{a_m}\right)}R\right]+\frac{1}{2}\sqrt{\left(\frac{s}{a_q}\right)}R\cosh\sqrt{\left(\frac{s}{a_q}\right)}R\left[\sqrt{\left(\frac{s}{a_m}\right)}\times\right.$$

$$\left.\times R\cosh\sqrt{\left(\frac{s}{a_m}\right)}R-\sinh\sqrt{\left(\frac{s}{a_m}\right)}R\right]+\frac{1}{2}\sqrt{\left(\frac{s}{a_m}\right)}R\sqrt{\left(\frac{s}{a_m}\right)}\times$$

$$\times R\sinh\sqrt{\left(\frac{s}{a_m}\right)}R\sinh\sqrt{\left(\frac{s}{a_q}\right)}R.$$

To obtain the first term of (2.8) we expand the numerator and denominator $\varphi(s)/\psi(s)$ in series, divide the fraction by $\sqrt{s/a_m}\times\sqrt{s/a_q}$ and afterwards differentiate the denominator with respect to s, and take the limit of this expression as $s\to 0$. As a result we obtain

$$\lim_{s\to 0}\frac{\varphi(s)}{\psi'(s)}=0.$$

For the other terms, calculation gives

$$\varphi(s_n)=a_m\sin\mu_n\frac{r}{R}\left[\sin\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n-\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n\cos\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n\right]+$$

$$+a_q\mu_n\cos\mu_n\sin\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n\frac{r}{R},$$

$$\varphi(s_l)=-\left[\sin\sqrt{\left(\frac{a_m}{a_q}\right)}\mu_l-\sqrt{\left(\frac{a_m}{a_q}\right)}\mu_l\cos\sqrt{\left(\frac{a_m}{a_q}\right)}\mu_l\right]a_q\sin\mu_l\frac{r}{R},$$

$$\psi'(s_n)=-\frac{1}{2}\mu_n\cos\mu_n\left[\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n\cos\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n-\right.$$

$$\left.-\sin\sqrt{\left(\frac{a_q}{a_m}\right)}\mu_n\right],$$

$$\psi'(s_l)=\frac{1}{2}\mu_l^2\sin\mu_l\sin\sqrt{\left(\frac{a_m}{a_q}\right)}\mu_l.$$

Hence, the distribution function of the transfer potential of mass in the sphere (2.6) can be written, after a few trans-

formations, thus:

$$\frac{\theta(r,\tau)-\theta_0}{\theta_0}=\frac{2\,\mathrm{Lu}\,\mathrm{Pn}_1}{\mathrm{Lu}-1}\times$$

$$\times\left\{\sum_{n=1}^{\infty}\frac{\exp\left(-n^2\pi^2\,\mathrm{Fo}\right)}{(-1)^n\,n\pi\left(\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}-\frac{n\pi}{\sqrt{\mathrm{Lu}}}\cos\frac{n\pi}{\sqrt{\mathrm{Lu}}}\right)}\times\right.$$

$$\times\left[\left(\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}-\frac{n\pi}{\sqrt{\mathrm{Lu}}}\cos\frac{n\pi}{\sqrt{\mathrm{Lu}}}\right)\frac{\sin n\pi\frac{r}{R}}{\frac{r}{R}}+\right.$$

$$\left.+(-1)^n\frac{n\pi}{\sqrt{\mathrm{Lu}}}\,\frac{\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}\,\frac{r}{R}}{\frac{r}{R}}\right]-$$

$$-\sum_{l=1}^{\infty}\frac{\left[\sin\sqrt{(\mathrm{Lu})}\,\mu_l-\sqrt{(\mathrm{Lu})}\,\mu_l\cos\sqrt{(\mathrm{Lu})}\,\mu_l\right]}{\mathrm{Lu}\,\mu_l^2\sin\mu_l\sin\sqrt{(\mathrm{Lu})}\,\mu_l}\times$$

$$\left.\times\frac{\sin\mu_l\frac{r}{R}}{\frac{r}{R}}\exp\left(-\mu_l^2\,\mathrm{Lu}\,\mathrm{Fo}\right)\right\},\qquad(2.9)$$

where μ_l are the roots of the characteristic equation (2.7): their values are given in Table 34.

TABLE 34. ROOTS μ OF THE EQUATION $\tan\mu=\mu$.

l	μ_l
1	0·0000
2	4·4934
3	7·7253
4	10·9041
5	14·0662
6	17·2208

(b) *Heat and Mass Transfer in an Infinite Cylinder* ($\Gamma = 1$, $\xi = r$, $\varepsilon = 0$, $t(r, 0) = t_0 = \text{const.}$, $\theta(r, 0) = \theta_0 = \text{const.}$) The relative temperature and the non-dimensional transfer potential of mass at any point in an infinite cylinder have respectively the forms

$$\frac{t(r,\tau)}{t_0} = \sum_{n=1}^{\infty} \frac{2}{\mu_n J_1(\mu_n)} J_0\left(\mu_n \frac{r}{R}\right) \exp\left(-\mu_n^2 \mathrm{Fo}\right), \quad (2.10)$$

$$\frac{\theta(r,\tau) - \theta_0}{\theta_0} = \frac{2\,\mathrm{Lu}\,\mathrm{Pn}_1}{\mathrm{Lu} - 1}\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\mathrm{Lu}}} \times \right.\right.$$

$$\times \frac{J_0\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}} \frac{r}{R}\right)}{\mu_n J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)} - \frac{J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n J_1(\mu_n)}\Bigg] \exp\left(-\mu_n^2 \mathrm{Fo}\right) -$$

$$- \sum_{i=1}^{\infty} \frac{J_1[\sqrt{(\mathrm{Lu})}\,\mu_i]}{\sqrt{(\mathrm{Lu})}\,\mu_i J_0[\sqrt{(\mathrm{Lu})}\,\mu_i]\,J_0'(\mu_i)} J_0\left(\mu_i \frac{r}{R}\right) \times$$

$$\left.\times \exp\left(-\mu_i^2\,\mathrm{Lu}\,\mathrm{Fo}\right)\right\}, \quad (2.11)$$

where μ_n are the roots of the characteristic equation

$$J_0(\mu) = 0\,,$$

and μ_i are the roots of the characteristic equation

$$J_1(\mu) = 0\,.$$

The values of the characteristic roots μ_n and μ_i are shown in Table 35.

TABLE 35. ROOTS OF THE CHARACTERISTIC EQUATIONS $J_0(\mu) = 0$ and $J(\mu) = 0$

n or l	roots μ_n of equation $J_0(\mu) = 0$	roots μ_l of equation $J_1(\mu) = 0$
1	2·4048	3·8317
2	5·5201	7·0156
3	8·6537	10·1735
4	11·7915	13·3237
5	14·9309	16·4706
6	18·0711	19·6159
7	21·2116	22·7601
8	24·3525	25·9037
9	27·4935	29·0468
10	30·6346	32·1897

(c) *Heat and Mass Transfer in an Infinite Plate* ($\Gamma = 0$, $\xi = x$, $\varepsilon = 0$, $t(x, 0) = t_0 = \text{const.}$, $\theta(x, 0) = \theta_0 = \text{const.}$). The distribution of transfer potentials of heat and mass has the form

$$\frac{t(x,\tau)}{t_0} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\cos(2n-1)\frac{\pi}{2}\frac{x}{R}}{(2n-1)\frac{\pi}{2}} \times$$

$$\times \exp\left[-\frac{(2n-1)^2\pi^2}{4}\mathrm{Fo}\right], \qquad (2.12)$$

$$\frac{\theta(x,\tau)-\theta_0}{\theta_0} = \frac{2\,\mathrm{Lu}\,\mathrm{Pn}_1}{\mathrm{Lu}-1} \times$$

$$\times \left\{ \sum_{n=1}^{\infty} \frac{\exp\left[-\frac{(2n-1)^2\pi^2}{4}\mathrm{Fo}\right]}{(2n-1)\frac{\pi}{2}} \times \right.$$

$$\left. (-1)^{n+1}(2n-1)\frac{\pi}{2}\sin\frac{(2n-1)\frac{\pi}{2}}{\sqrt{\mathrm{Lu}}} \right.$$

$$\times \left[(-1)^{n+1}\frac{1}{\sqrt{\mathrm{Lu}}}\cos\frac{(2n-1)\frac{\pi}{2}\frac{x}{R}}{\sqrt{\mathrm{Lu}}} - \right.$$

$$\left. - \sin\frac{(2n-1)\frac{\pi}{2}}{\sqrt{\mathrm{Lu}}}\cos(2n-1)\frac{\pi}{2}\frac{x}{R} \right] -$$

$$- \sum_{l=1}^{\infty} \frac{\frac{1}{\sqrt{\mathrm{Lu}}} \sin \sqrt{(\mathrm{Lu})}\, \pi l}{(-1)^l \pi l \cos \sqrt{(\mathrm{Lu})}\, \pi l} \times$$

$$\times \cos \pi l \frac{x}{R} \exp\left(-\pi^2 l^2 \mathrm{Lu}\ \mathrm{Fo}\right) \Bigg\}, \qquad (2.13)$$

where $n = 1, 2, 3 \ldots$, $l = 1, 2, 3, \ldots$,

(d) *Heat and Mass Transfer in a Finite Rod* ($\Gamma = 0$, $\xi = x$, $t(x, 0) = t_0 = \text{const}$, $\theta(x, 0) = \theta_0 = \text{const}$). We will suppose that one end of the rod which is at the origin of coordinates ($x = 0$) has, at the time of the experiment, a constant temperature equal to the temperature of the heater t_H,

$$t(0, \tau) = t_H = \text{const}. \qquad (2.14)$$

The other end of the rod ($x = l$) remains for the duration of the experiment at the temperature of the surrounding medium, i.e.

$$t(l, \tau) = t_c = t_0 = \text{const}, \qquad (2.15)$$

where $t_H > t_0$. The side surfaces of the rod are insulated from the transfer of heat and mass.

The general solution of the system of equations (I), (II), under constant initial conditions (VI) and boundary conditions (VII), (2.14) and (2.15), has the form

$$\frac{t(x, \tau) - t_0}{t_H - t_0} = \left(1 - \frac{x}{l}\right) +$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{2}{\pi n} \sin \pi n \left(1 - \frac{x}{l}\right) \exp\left(-\pi^2 n^2 \mathrm{Fo}_1\right), \quad (2.16)$$

$$\frac{\theta(x, \tau) - \theta_0}{\theta_0} = \mathrm{Pn}_2 \left(\frac{x}{l} - \frac{1}{2}\right) + \frac{2\mathrm{Lu}\ \mathrm{Pn}_2}{\mathrm{Lu} - 1} \times$$

$$\times \left\{ \sum_{n=1}^{\infty} \frac{\exp\left(-n^2 \pi^2 \mathrm{Fo}_1\right)}{n\pi \sin \frac{n\pi}{\sqrt{\mathrm{Lu}}}} \left\{ \frac{(-1)^n}{\sqrt{\mathrm{Lu}}} \cos \frac{n\pi}{\sqrt{\mathrm{Lu}}} \frac{x}{l} + \right. \right.$$

$$\left. \left. + \sin \frac{n\pi}{\sqrt{\mathrm{Lu}}} \sin n\pi \frac{x}{l} - \frac{1}{\sqrt{\mathrm{Lu}}} \times \right. \right.$$

$$\times \cos\left(1-\frac{x}{l}\right)\frac{n\pi}{\sqrt{\mathrm{Lu}}}\Big] + \sum_{k=1}^{\infty}\frac{(-1)^k-\cos\sqrt{(\mathrm{Lu})}\,\pi k}{\sqrt{(\mathrm{Lu})}\,\pi k\sin\sqrt{(\mathrm{Lu})}\,\pi k}\times$$

$$\times\cos k\pi\frac{x}{l}\exp\left(-k^2\pi^2\,\mathrm{Lu}\,\mathrm{Fo}_1\right)\Bigg\}. \qquad (2.17)$$

In the equations (2.16) and (2.17) $n = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$, $\mathrm{Pn}_2 = \delta(t_H - t_0)/\theta_0$ is the Posnov criterion, $\mathrm{Fo}_1 = a_q\,\tau/l^2$.

The expressions (2.16) and (2.17) can be used conveniently at great values of Fo_1, i.e. when $\mathrm{Fo}_1 > 0{\cdot}1$, in which case it is possible to limit ourselves to one or two terms of the infinite series for calculating the transfer potentials. At small values of Fo_1 a correct result is obtained only if several terms of the expansion are used, and in this case it is better to employ a more convenient (i.e. rapidly converging) expression:

$$\frac{t(x,\tau)-t_0}{t_H-t_0} = \operatorname{erfc}\frac{l}{2\sqrt{\mathrm{Fo}_1}} + \sum_{n=1}^{\infty}\left\{\operatorname{erfc}\frac{2n+\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}} - \operatorname{erfc}\frac{2n-\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}}\right\}, \qquad (2.18)$$

$$\frac{\theta(x,\tau)-\theta_0}{\theta_0} = \frac{\mathrm{Lu}\,\mathrm{Pn}_2}{\mathrm{Lu}-1}\Bigg\{\frac{1}{\sqrt{\mathrm{Lu}}}\operatorname{erfc}\frac{\frac{x}{l}}{2\sqrt{(\mathrm{Lu}\,\mathrm{Fo}_1)}} +$$

$$+\frac{1}{\sqrt{\mathrm{Lu}}}\sum_{k=1}^{\infty}\left[\operatorname{erfc}\frac{2k+\frac{x}{l}}{2\sqrt{(\mathrm{Lu}\,\mathrm{Fo}_1)}} + \operatorname{erfc}\frac{2k-\frac{x}{l}}{2\sqrt{(\mathrm{Lu}\,\mathrm{Fo}_1)}}\right] -$$

$$-\operatorname{erfc}\frac{\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}} - \sum_{n=1}^{\infty}\left[\operatorname{erfc}\frac{2n+\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}} - \operatorname{erfc}\frac{2n-\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}}\right] +$$

$$+\frac{2}{\sqrt{\mathrm{Lu}}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^n\left[\operatorname{erfc}\frac{n+\frac{k+\frac{x}{l}}{\sqrt{\mathrm{Lu}}}}{2\sqrt{\mathrm{Fo}_1}} + \operatorname{erfc}\frac{n-\frac{k-\frac{x}{l}}{\sqrt{\mathrm{Lu}}}}{2\sqrt{\mathrm{Fo}_1}}\right] +$$

$$+\frac{2}{\sqrt{\mathrm{Lu}}}\sum_{n=1}^{\infty}\operatorname{erfc}\frac{2n+\dfrac{\dfrac{x}{l}}{\sqrt{\mathrm{Lu}}}}{2\sqrt{\mathrm{Fo}}}\Bigg\}. \qquad (2.19)$$

(c) *Analysis of the solutions*. The temperature field is analogous to the temperature field of the "dry" material. The relative temperature of the body is a function of the criterion of Fourier (Fo) and of the relative coordinate ξ/R:

$$\frac{t(\xi,\tau)}{t_0}=f\left(\mathrm{Fo},\frac{\xi}{R}\right).$$

Since the characteristic roots increase from the point where $\mathrm{Fo}>\mathrm{Fo}_1$ (i.e. during the stage where regular conditions have been established), we can neglect all terms of the series except the first. In this case the ratio between the non-dimensional temperature and the time can be written thus:

$$\frac{t(\xi,\tau)}{t_0}=A_1\,\Phi\left(\mu_1,\frac{\xi}{R}\right)\exp\left(-\mu_1^2\,\mathrm{Fo}\right).$$

Here A_1 is the initial thermal amplitude. For an infinite plate $A_1=4/\pi$, for a sphere $A_1=2/\pi$, and for an infinite cylinder

$$A_1=\frac{2}{2.405\,I_1(2.405)},$$

$\Phi(\mu_1,\xi/R)$ is a function which describes the distribution of temperature over the characteristic coordinate ξ/R, μ_1 is the first root of the characteristic equation (for an infinite plate $\mu_1=\pi/2$, for a sphere $\mu_1=\pi$, for an infinite cylinder $\mu_1=2{\cdot}405$), I_1 is a modified Bessel function of the first kind and the first order.

In the stationary state, i.e. for $\mathrm{Fo}\to\infty$, the non-dimensional temperature for the plate, the cylinder, and the sphere, becomes equal to 0, and, for a rod of limited length, $\dfrac{t(x,\tau)-t_0}{t_{II}-t_0}=1-x/l$, i.e. is a linear function of x/l. The field of the non-dimensional potential of transfer of mass is characterized not only by Fo and x/R, but also by the criteria Lu

and Pn. The analytical solutions for $\theta(\xi, \tau)-\theta_0/\theta_0$ are the difference of two series, the first of which describes the modified temperature field and the second the modified field of potential of mass.

With increase in the non-dimensional time, the distribution of transfer potential of mass tends to a stationary state, in which for an infinite plate and an infinite cylinder, and also for a sphere, $\theta(\xi, \infty) = \theta_0$, and for a finite rod

$$\frac{\theta(x, \tau) - \theta_0}{\theta_0} = \mathrm{Pn}_2\left(\frac{x}{l} - \frac{1}{2}\right).$$

The values of $\Delta = [\theta(x, \tau) - \theta_0]/\theta_0 \mathrm{Pn}_2$, as calculated by Sizyakova for a finite rod, in relation to Fo, x/l and Lu are represented in Fig. 33. $[\theta(x, \tau)-\theta_0]/\theta_0$ was calculated from the expression (2.17). It follows from the curves that, under the influence of a changing temperature field, there arises in the moist

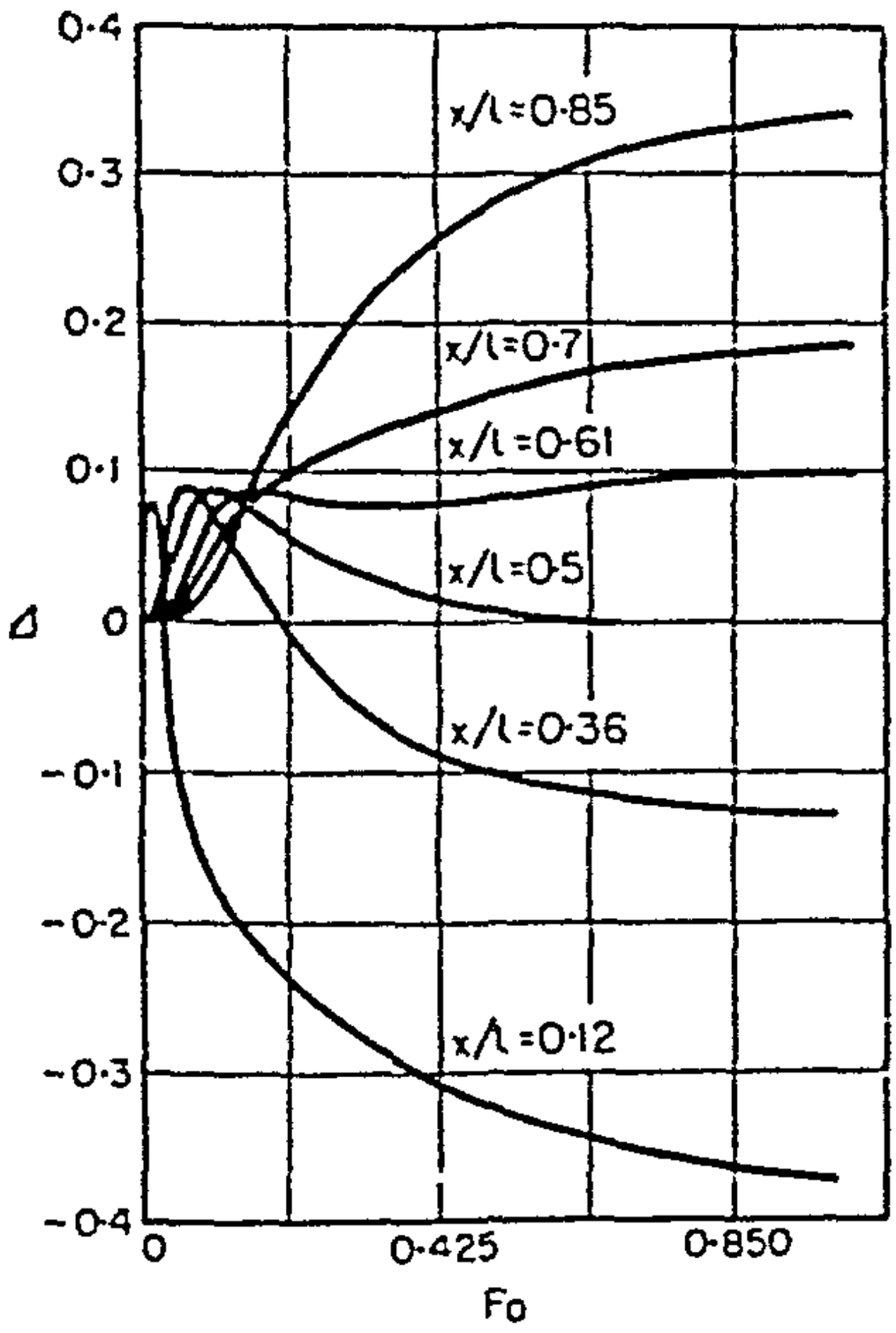


FIG. 33. Relation between $\Delta = [\theta(x, \tau) - \theta_0]/\theta_0 \mathrm{Pn}_2$ and the value of the Fourier criterion and the non-dimensional coordinate (Lu = 0·355)

material a wave of transfer potential of mass (mass-content) which is damped with respect to the time. Calculation shows that, for small values of Lu, the wave is damped very rapidly. With increasing Lu the wave penetrates more deeply into the material, spreads more rapidly, and has an increasingly large maximum.

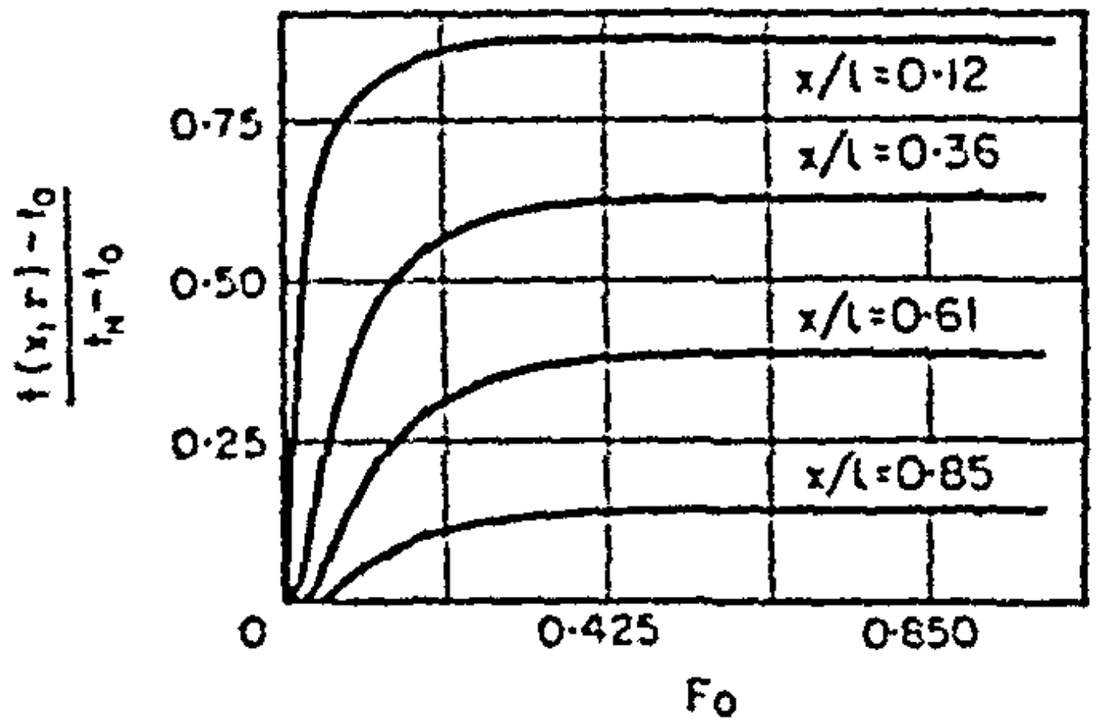


FIG. 34. Analytical curves of the change of the non-dimensional temperature with time

The rate of change of the potential of mass at each point with increasing Lu increases rapidly and the linear stationary distribution arrives earlier. For example, For Lu = 0·8 and Fo = 0·5 the distribution of moisture content even in the finite rod can be considered to be already stationary. The local temperature (Fig. 34) at first rapidly increases, and then slowly approximates to the constant value corresponding to the stationary linear distribution of temperature in the specimen. A comparison of the curves of change of the non-dimensional potentials of transfer at the same moments of time shows that the most rapid change of the transfer potential of mass occurs at the same time as the most rapid change in the transfer potential of heat. At small values of Lu the changes in θ proceed considerably more slowly than the changes in t. The stationary distribution of temperature occurs much earlier than the stationary distribution of transfer potential of mass. It also follows from calculation that the

of mass is carried out only in the liquid phase. However, these first attempts to use the analytical solutions of the system of equations (I), (II) may lead to the working out of new non-stationary methods for determining the coefficients of transfer.

(a) *Kuznetsov's method.* For the determination of the coefficients of heat and mass transfer, Kuznetsov makes use of a sample in the form of a sphere. His method makes it possible to determine, by one experiment, the mass diffusion coefficient a_m and the thermal gradient coefficient δ; at the same time the quantity a_q is found by an independent method, for example, by the method of the regular regime. To determine the coefficients it is necessary to know the distribution through the material of the temperature and the transfer potential of mass (or mass-content).

Kuznetsov's method is based on the following results of the analytical solution:

(a) The relationship between $\Delta = u(r_1\tau) - u_0/u_0 \mathrm{Pn}_1$ and the criterion Fo, as will be seen from Fig. 35, is a single-valued

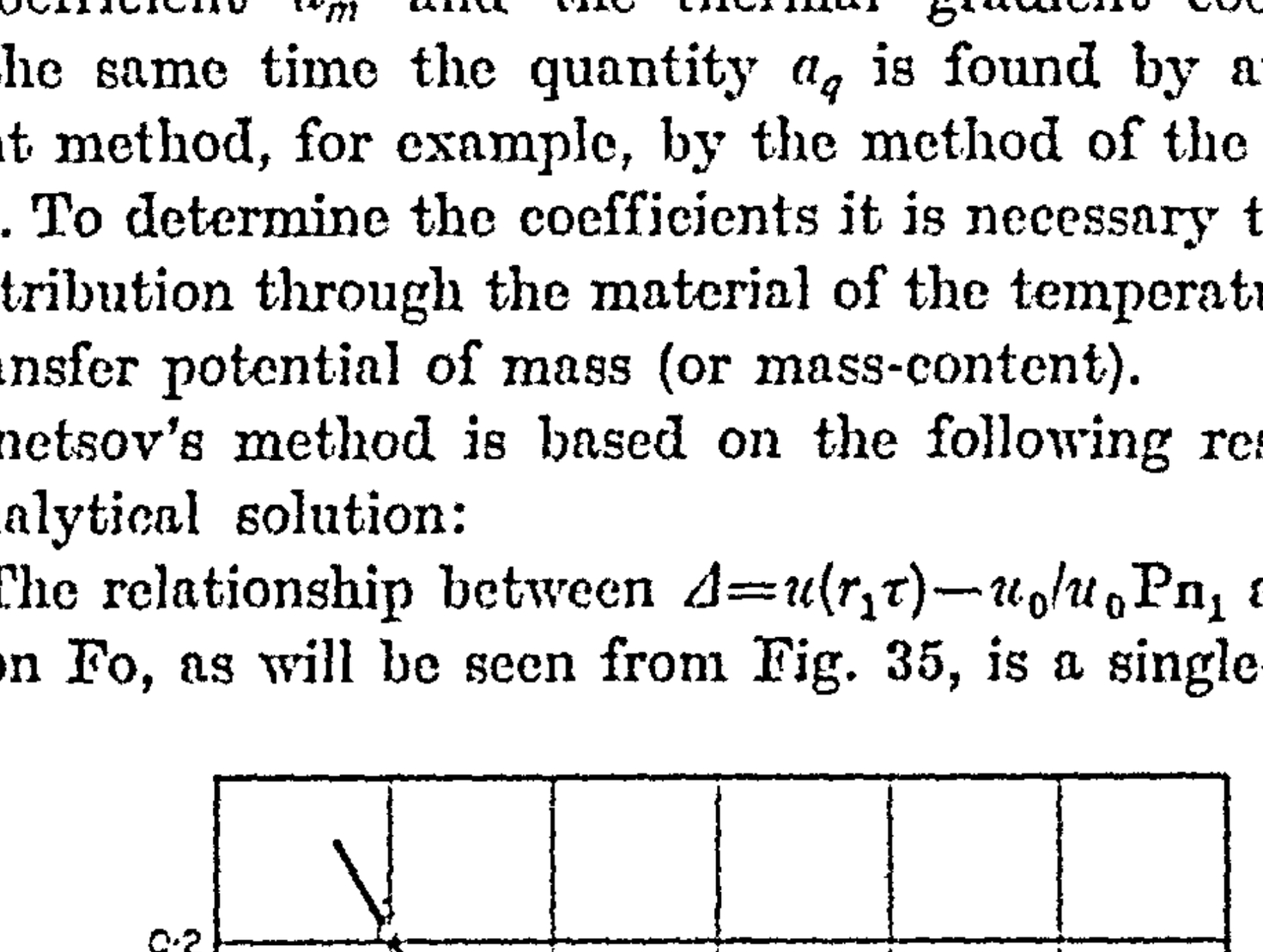


Fig. 35. Relation between $\frac{\bar{\theta}_m}{\mathrm{Pn}_1}$ and the Fourier criterion under the condition $0{\cdot}09 < \mathrm{Fo} < 0{\cdot}5$ and $0 < \mathrm{Lu} < 0{\cdot}5$: × — Lu = 0·2, ○ — Lu = 0·3, △ — Lu = 0·4.

function in the interval between Fo = 0·09 and Fo = 0·5, and between values of the criterion Lu from 0 to 0·5;

(b) For large values of Fo the distribution curve of transfer potential of mass in the material, intersects the straight line of the initial distribution in a definite point $(x/R)_A$ (Fig. 36).

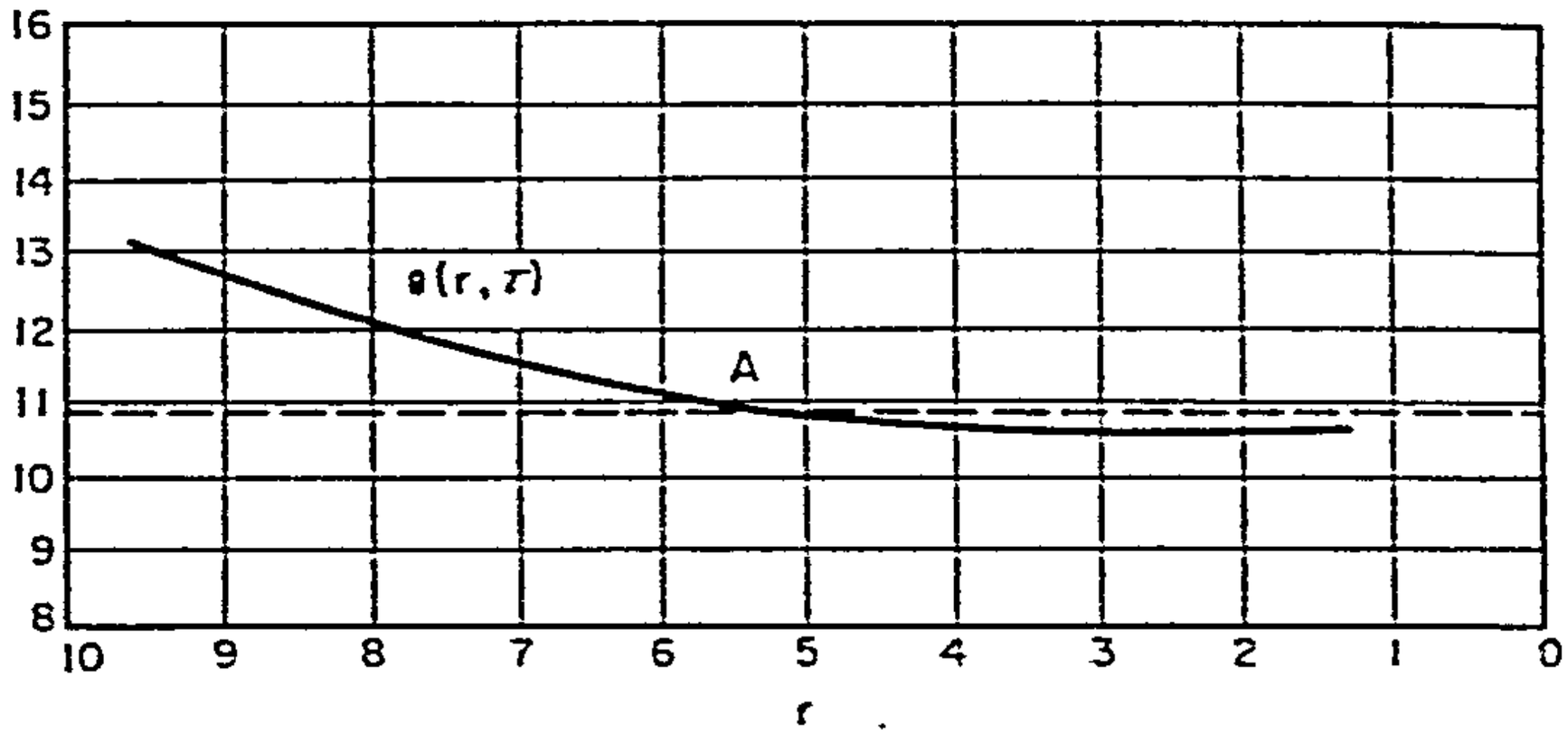


FIG. 36. Curve of distribution of mass content of sand along the radial coordinate r (cm) in the process of cooling

From Fig. 37 it follows that for values of the criterion Fo between 0·09 and 0·30 this point is a single-valued function of the criterion Lu.

Let us illustrate the method by calculating the coefficients of heat and mass transfer for sand with initial parameters of the transfer of mass $u_0 = 10{\cdot}9$ per cent and of heat $t = 54°C$. The radius of the flask into which the sand was poured was $R = 10$ cm. The coefficient of thermal diffusivity of sand $a_q = 2{\cdot}3 \times 10^{-3}$ m²/hr. Let $\tau = 25$ min. Then the Fourier criterion will be

$$\text{Fo} = \frac{a_q \tau}{R^2} = \frac{2.2 \times 10^{-3} \times 25}{1 \times 10^{-2} \times 60} = 0{\cdot}096.$$

From Fig. 35 we find that a value of $\Delta_R = \frac{u(R_1, \tau) - u_0}{u_0 \text{Pn}_1}$ $= 0{\cdot}212$ corresponds to the criterion Fo = 0·096. The value of the non-dimensional transfer potential of mass on the surface of the body $\frac{u(R, \tau) - u_0}{u_0}$ we find from its distribution

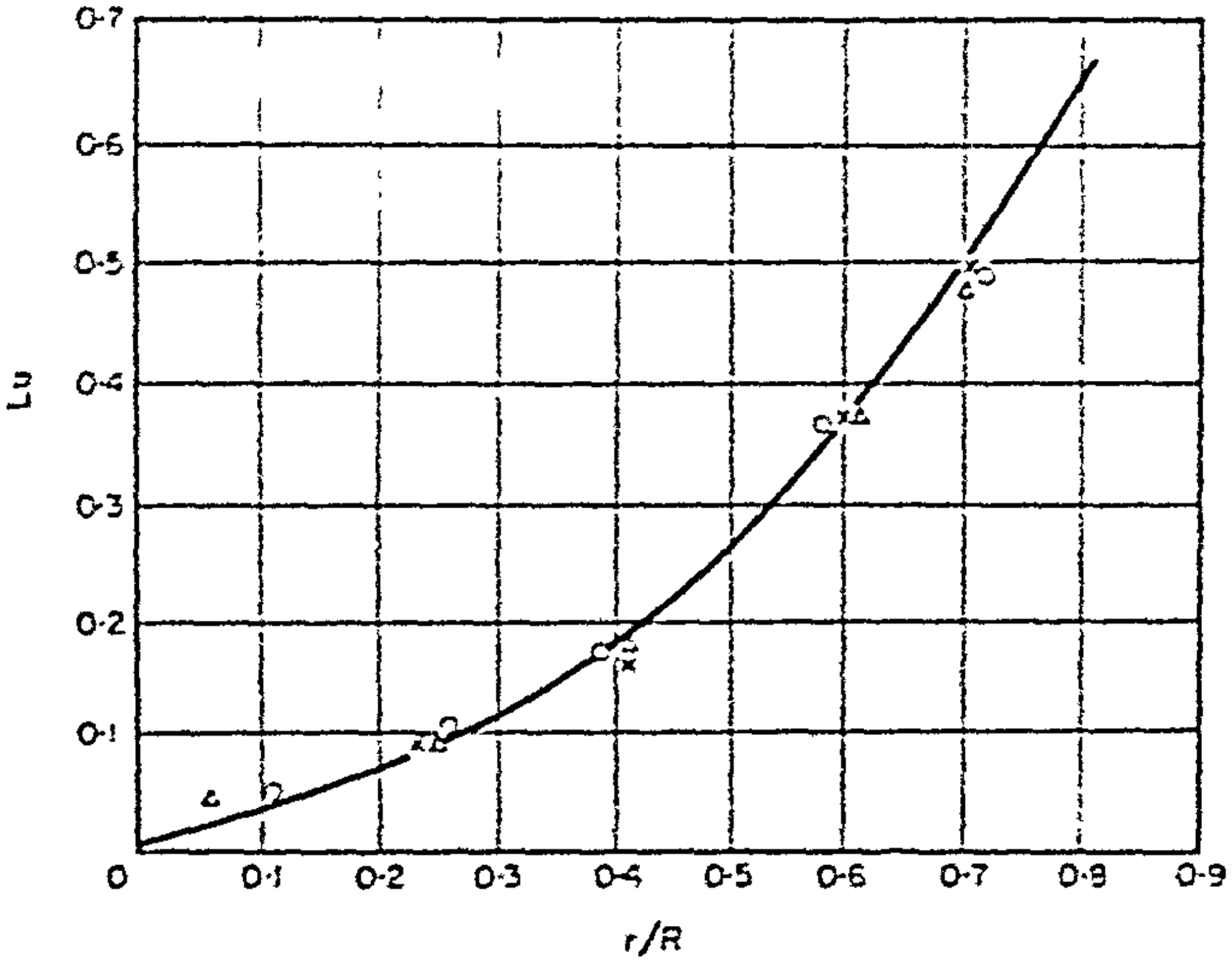


FIG. 37. Relation between the criterion Lu and the relative coordinate $\left(\frac{r}{R}\right)_A$, corresponding to the point of intersection of the curve of moisture distribution in the process of cooling with the line of the initial moisture content (meaning of symbols as in Fig. 35)

curve shown in Fig. 36:

$$\frac{u(R,\tau)-u_0}{u_0}=\frac{13{\cdot}3-10{\cdot}9}{10{\cdot}9}=0{\cdot}22.$$

The Posnov criterion is

$$\mathrm{Pn}_1=\frac{u(R,\tau)-u_0}{u_0\Delta_R}=\frac{0{\cdot}22}{0{\cdot}212}=1{\cdot}03\,.$$

Hence, the thermo-gradient coefficient is

$$\delta=\frac{u_0\mathrm{Pn}_1}{t_0}=\frac{0{\cdot}109\times1{\cdot}03}{54}=0{\cdot}207\times10^{-2}\ \mathrm{deg}^{-1}=0{\cdot}207\ \text{per cent/deg.}$$

We determine from Fig. 36 the relative coordinate $(r/R)_A$ of the intersection of the curve $u(r, \tau)$ with the line u_0. This is equal to 0·5. From Fig. 37 we find that this coordinate

corresponds to a value of Lu equal to 0·26; hence the coefficient is

$$a_m = \mathrm{Lu}\, a_q = 0.26 \times 2.3 \times 10^{-3} = 0.6 \times 10^{-3}\ \mathrm{m^2/hr}.$$

(b) *Sizyakova's method.* According to the analytical solution, the distribution of the transfer potential of mass with respect to time and with respect to the coordinates, as already pointed out above, has a maximum for small values of Fo. The value of this maximum for a given value of the criterion Lu is the same for all coordinates, but increases with increasing values of the criterion Lu. The nature of this change of the field of transfer potential of mass allowed Sizyakova to propose a new rapid method of determining the coefficients of heat and mass transfer from one experiment. For this purpose it is necessary to construct the curves of distribution of transfer potential of mass (or moisture content) and temperature in a rod for a short interval of time (about one hour) after the beginning of the experiment. The distribution curve of the transfer potential of mass θ or moisture content, will have a maximum $\theta_{\max}$ at a certain point with the non-dimensional coordinate x/l, which we will call $(x/l)_{\max}$. For the same point and the same time the temperature of the sample has the value $t[(x/l)_{\max},\ \tau_{\max}]$. Knowing this quantity, t_H and t_0 from the analytical solution (2.18), we can find the corresponding value of the Fourier criterion, which we will call $\mathrm{Fo}_{\max}$. This is easy to do since, for small values of Fo the first term of the expansion is sufficient, i.e.

$$\frac{t\left[\left(\frac{x}{l}\right)_{\max}, \tau_{\max}\right] - t_0}{t_H - t_0} = \operatorname{erfc} \frac{\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}}\,. \qquad (3.1)$$

Substituting on the lefthand side of (3.1) the known values, and making use of tables of the value of the error-integral, we find $\mathrm{Fo}_{\max}$, whence the coefficient of temperature conductivity will be

$$a_q = \frac{\mathrm{Fo}_{\max} l^2}{\tau_{\max}}\,.$$

Calculations show that the relation of Fo_{max} to ln Lu for each coordinate x/l is linear. Knowing Fo_{max} and the coordinate $(x/l)_{max}$ from Fig. 38 it is possible to find the value of

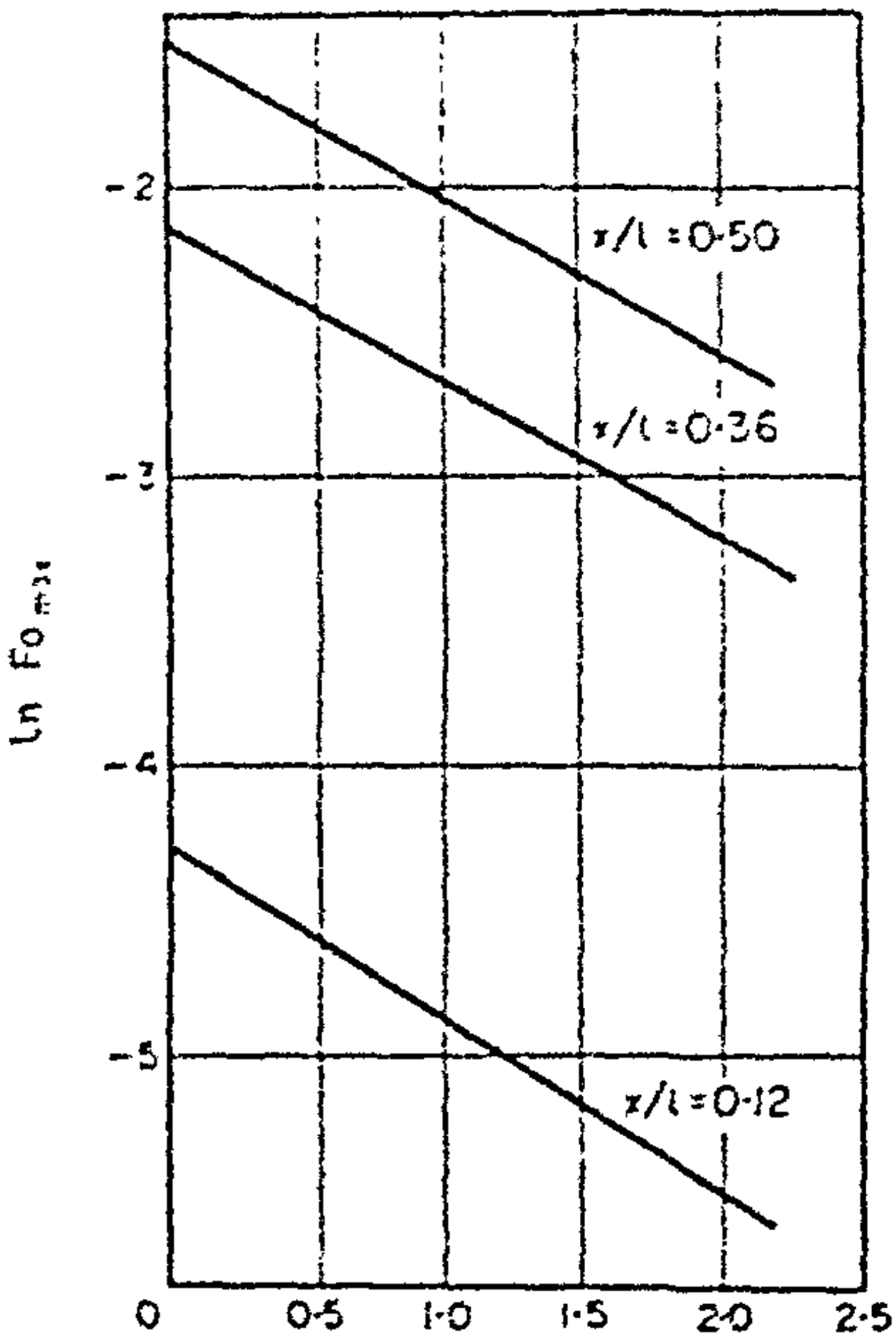


FIG. 38. Relation between ln Fo_{max} (axis of ordinates) and ln Lu (axis of abscissae)

ln Lu, and then the value of the criterion Lu. The coefficient of potential conductivity will then be

$$a_m = a_q \mathrm{Lu}.$$

In the analysis of the relations of Δ it was shown that Δ_{max} does not depend on the coordinates or on the criterion Fo, but depends only on the criterion Lu. This renders it possible to find the value of the Posnov criterion (Pn_1) and, consequently, the thermogradient coefficient δ. Thus we can find Δ_{max} from Fig. 39 on the basis of the already defined value of the criterion Lu. Hence we determine the Posnov criterion:

$$\mathrm{Pn}_1 = \frac{\theta_{max} - \theta_0}{\theta^0 \Delta_{max}}.$$

Then the thermo-gradient coefficient is

$$\delta = \frac{c_m \, \mathrm{Pn}_1 \, \theta_0}{t_H - t_0} \doteq \frac{\mathrm{Pn}_1 \, u_0}{t_H - t_0} \, .$$

Sizyakova's method makes it possible to determine from one experiment the coefficients a_m, a_q and δ. In this respect

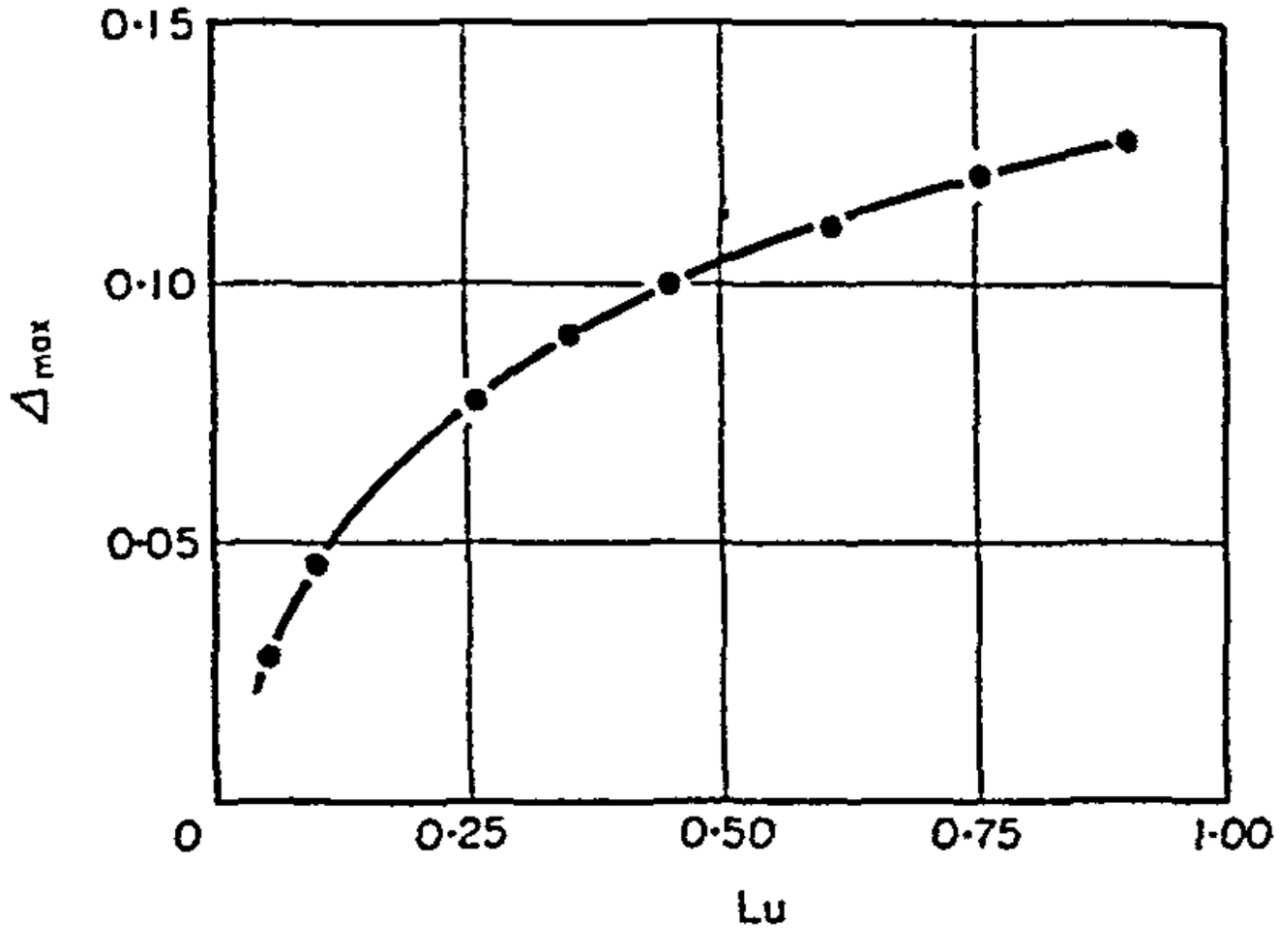


FIG. 39. Graph of Δ_{max} against Lu

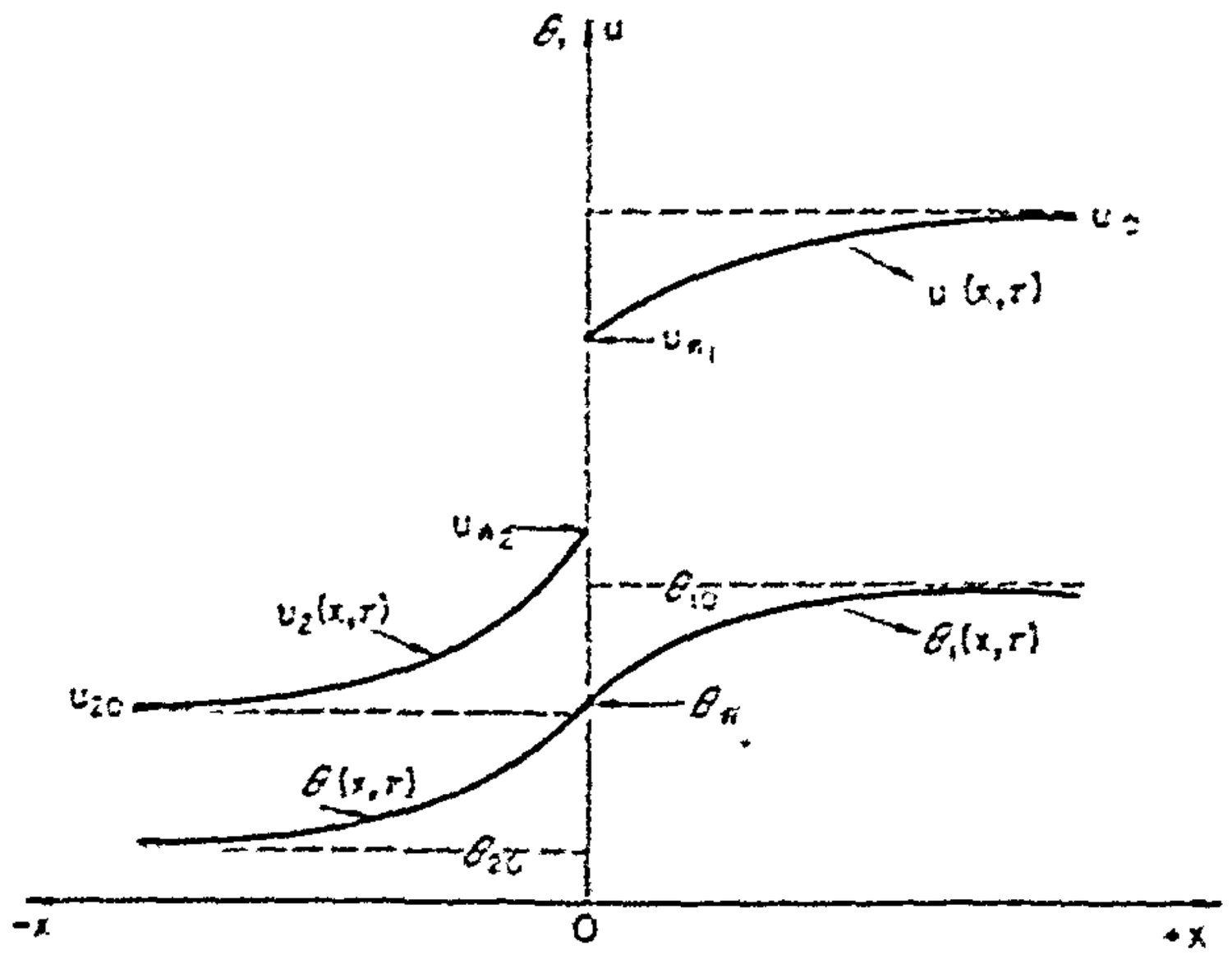
it is more convenient than Kuznetsov's method, but, like Kuznetsov's method, it is limited to low temperatures of the material (below 50°C) and to a narrow interval of values of the criterion Fo. The further development of non-stationary methods of determining the coefficients of heat and mass transfer must be placed in the first rank of these questions

(c) *Luikov's method*. The coefficients of mass transfer a_m and λ_m, and the thermodynamical characteristics θ and c_m can be determined from the configuration of the non-stationary field of mass content u of the body under investigation, by placing this in contact with a standard capillary-porous body (contact method).

The theory of this method is based on the solution of the problem of mass transfer for a system of two semi-infinite bodies placed in contact. The problem may be stated mathematically thus: Two semi-infinite bodies with different potentials θ_{10} and θ_{20} are placed with their free surfaces in

contact. The boundary conditions may be written as follows: (see Fig. 40).

$$\theta_1(x, 0) = \theta_{10}\,;\ \theta_2(x, 0) = \theta_{20}\,;\ \theta_1(0, \tau) = \theta_2(0, \tau) \quad (3.2)$$



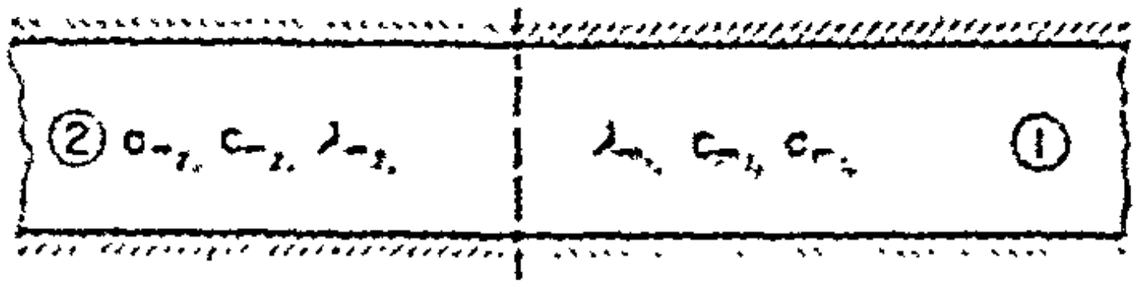


FIG. 40. Distribution of mass transfer potential θ and specific mass content u in a system of two semi-infinite bodies

The last equality denotes the equality of potentials at the boundary of contact at any moment of time.

$$-\frac{\partial \vartheta_1(0, \tau)}{\partial x} = \frac{\lambda_{m2}}{\lambda_{m1}} \frac{\partial \vartheta_2(0, \tau)}{\partial x}\,;\ \frac{\partial \vartheta_1(+\infty, \tau)}{\partial x} = \frac{\partial \vartheta_2(-\infty, \tau)}{\partial x} = 0. \quad (3.3)$$

These boundary conditions correspond to the case where the body under investigation and the standard body are of infinite extent. It is known, however, from the theory of transfer, that the solution of this problem can be applied to bodies of finite dimensions, if the time of mass transfer is small

enough. It is important that the potentials $\theta_1(x, \tau)$ and $\theta_2(x, \tau)$ at distant points from the surface of contact should not change during the experiment, i.e.

$$\text{for } \tau < \tau_1, \qquad \frac{\partial \theta_1(+l, \tau)}{\partial x} = \frac{\partial \theta_2(-l, \tau)}{\partial x} \tag{3,3'}$$

where l is the length of the component cylinders, whose side surfaces must be insulated for moisture.

The solution of this problem has the form.

$$\frac{\theta_1(x, \tau) - \theta_{20}}{\theta_{10} - \theta_{20}} = \frac{k}{1+k}\left(1 + k \operatorname{erf} \frac{x}{2\sqrt{(a_{m1}\tau)}}\right), \tag{3.4}$$

$$\frac{\theta_2(x, \tau) - \theta_{20}}{\theta_{10} - \theta_{20}} = \frac{k}{1+k} \operatorname{erf} c \frac{|x|}{2\sqrt{(a_{m2}\tau)}}, \tag{3.5}$$

where k is the ratio of the criterion of mass activity of the investigated body to that of the standard body, equal to

$$k = \frac{\lambda_{m1}}{\lambda_{m2}} \sqrt{\left(\frac{a_{m2}}{a_{m1}}\right)} = \sqrt{\left(\frac{\lambda_{m1} c_{m1} \gamma_{01}}{\lambda_{m2} c_{m2} \gamma_0}\right)}. \tag{3.6}$$

From the solutions of (3.4) and (3.5) it follows that, at the boundary of contact ($x = 0$), a definite potential is established, θ_w, which remains constant during the whole process of mass transfer, i.e.

$$\frac{\theta_W - \theta_{20}}{\theta_{10} - \theta_{20}} = \frac{k}{1+k} = \text{const}. \tag{3.7}$$

Consequently, the solutions (3.4) and (3.5) can be written thus:

$$\frac{\theta_1(x_1 \tau) - \theta_W}{\theta_{10} - \theta_{20}} = \operatorname{erf} \frac{x}{2\sqrt{(a_{m1}\tau)}}, \tag{3.8}$$

$$\frac{\theta_2(x_1 \tau) - \theta_{20}}{\theta_W - \theta_{20}} = \operatorname{erf} \frac{|x|}{2\sqrt{(a_{m2}\tau)}}. \tag{3.9}$$

Let us pass from the potentials θ to the specific mass content. At the surface of contact there will be a discontinuity of the mass content, which will be maintained for the duration of the experiment (see Fig. 40).

$$\theta_W = \frac{u_2(0, \tau)}{c_{m2}} = \frac{u_1(x, \tau)}{c_{m1}}. \tag{3.10}$$

The specific mass capacity of the standard body (subscript 2) is taken equal to the 100th part of the maximum sorption moisture content (0·01 u_c). Thus, the maximum sorption mass content (moisture content) of the standard body corresponds to 100 units of the transfer potential of moisture. Then the potential θ_w will be equal to

$$\theta_W = \frac{u_2(0, \tau)}{u_c} \cdot 100 . \tag{3.11}$$

For small drops of the potential θ inside the body under investigation, the solution (3.8) can be written thus:

$$\frac{u_1(x, \tau) - u_{W1}}{u_{10} - u_{W1}} = \operatorname{erf} \frac{x}{2\sqrt{(a_m \tau)}}, \tag{3.12}$$

where u_{10} and u_{W1} are respectively the specific mass contents of the body under investigation before the experiment ($\tau = 0$) and on the surface of contact ($x = 0$).

If we denote the argument of the function erf by erf^{-1}, then we can write:

$$\operatorname{erf}^{-1} \left[\frac{u_1(x, \tau) - u_{W1}}{u_{10} - u_{W1}} \right] = \frac{x}{2\sqrt{(a_m \tau)}} . \tag{3.13}$$

Plotting the arguments of the function erf as ordinates, and the experimental points u_1 (x, τ), as abscissae — the coordinates of x — we obtain a straight line, which passes through the origin of coordinates. The tangent of the angle of slope of this line is equal to

$$\tan \varphi = \frac{1}{2\sqrt{(a_m \tau)}},$$

whence the coefficient a_m is obtained as

$$a_m = \frac{1}{4 \tau \tan^2 \varphi} . \tag{3.14}$$

To explain the method of calculation we will take an example from the curves of distribution of moisture content in a system of bodies in contact consisting of clay and paper, obtained 1·5 hr after the beginning of the mass exchange.

The moisture content of the standard body (filter paper) at the surface of contact was $u_{W2} = 79$ per cent. The moisture content of the body under investigation (clay) at the surface of contact was $u_{W1} = 28$ per cent.

The moisture content at points remote from the surface of contact of both bodies remained constant and equal to the initial moisture content, which shows that the condition (3.3) was observed. They were equal to $u_{10} = 32{\cdot}8$ per cent, $u_{20} = 11$ per cent. The potential θ_W was

$$\theta_W = \frac{u_2(0, \tau)}{u_c} \cdot 100 = \frac{79{\cdot}0}{22{\cdot}7} \cdot 100 = 348 \text{ units of potential,}$$

where u_c is the maximum moisture content of the filter paper at 40°C, equal to 22·7 per cent.

Let us calculate the ratio $\frac{u_1(x, \tau) - u_{W1}}{u_{10} - u_{W1}}$ and find the value of the argument of the function erf for values of the function equal to the given ratio. The results of this calculation are represented in Fig. 41 in the form of a graph, from which it will be seen that the experimental points lie on a straight line. The tangent of the angle of slope of the straight line, $\tan \psi = 0{\cdot}60 \text{ cm}^{-1} = 60 \text{ m}^{-1}$.

The coefficient of potential conductivity of mass transfer is equal to

$$a_{m1} = \frac{1}{4\tau \tan^2 \psi} = \frac{1}{1{\cdot}5 \times 4 \times (60)^2} = 4{\cdot}6 \times 10^{-5} \text{ m}^2/\text{hr}.$$

This value of the coefficient a_{m1} corresponds to the mass content of the body (clay) $u_{W1} = 0{\cdot}28$ kg/kg or to a potential of transfer of matter $\theta_W = 348$ units of potential.

The specific mass capacity c_{m1} is determined from the experimental curve $u_1 = f(\theta)$, obtained on the basis of experiments carried out with different initial mass-content (see Fig 42). From Fig. 42 we determine the specific mass capacity for clay from the relation

$$c_{m1} = \left(\frac{\partial u_1}{\partial \theta}\right)_\tau = 0 \cdot 03 \times 10^{-2} \text{ kg/kg units of potential.}$$

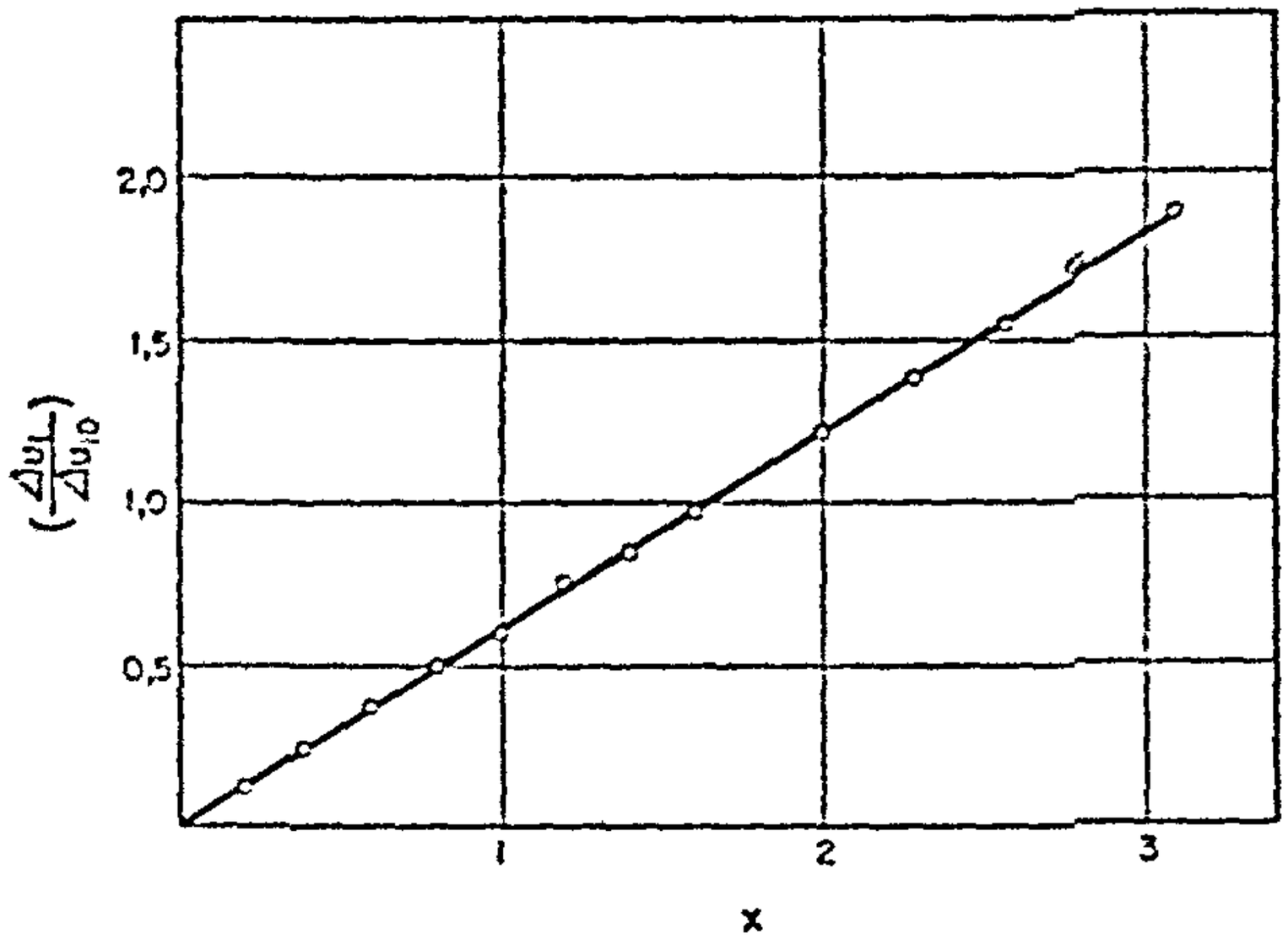


FIG. 41. Calculation of the coefficient a_m

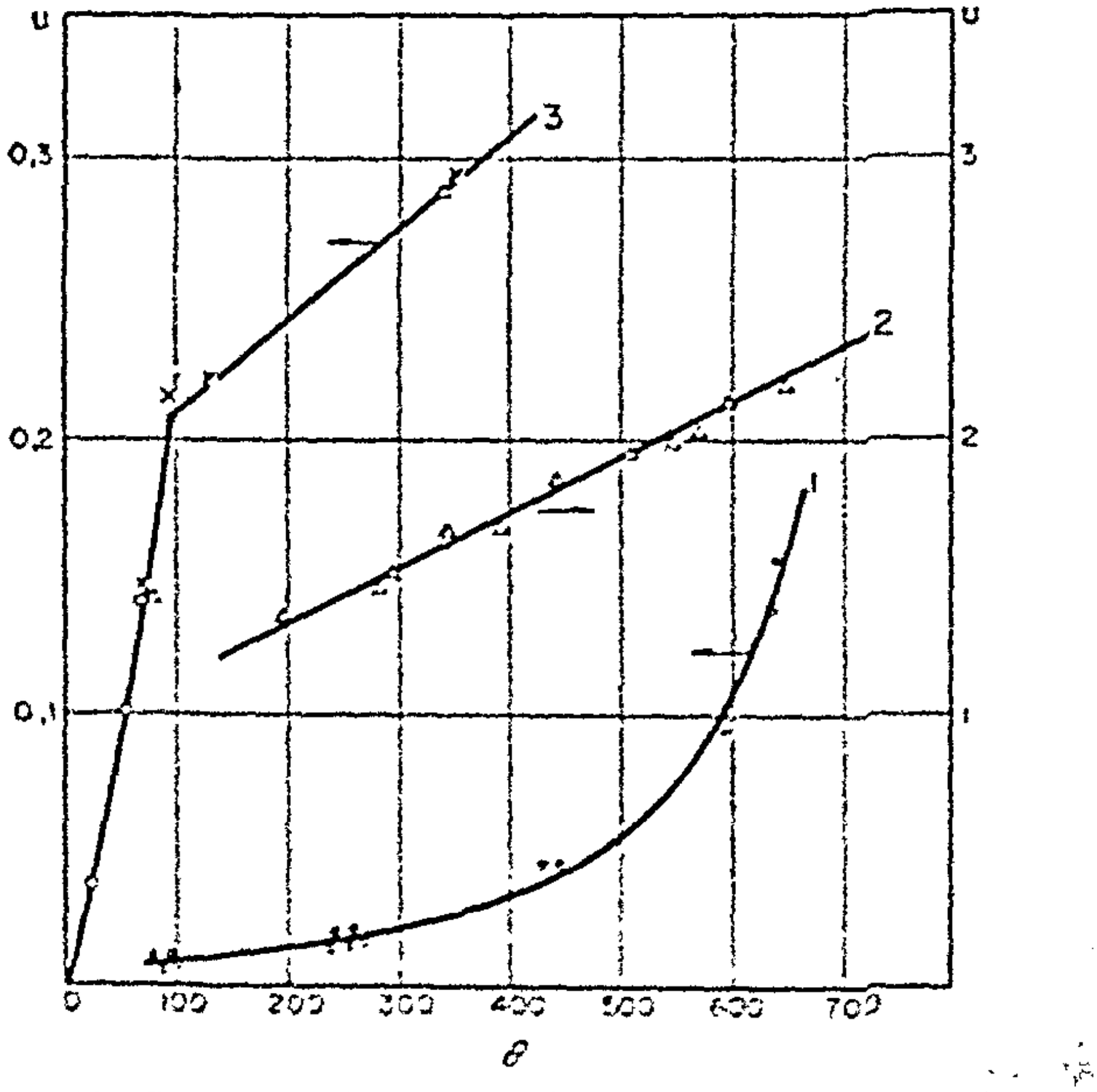


FIG. 42. Relation between the specific mass content and the potential of mass transfer for quartz sand (curve 1), wood (pine), (curve 2), and clay (curve 3)

content of which is determined by drying in a thermostat to constant weight. However, F. M. Polonskaya has recently worked out an apparatus in which the moisture content of the standard body under investigation is determined by special instruments. These instruments consist of two parallel threads, distant one from the other by a small distance r_0 from 4 to 6 mm. One of the threads acts as a source of heat, being heated by an electric current for a short interval of time τ_0 (from 15 to 30 sec), so that the increase of temperature near the thread is not more than 3 — 4°C. The second thread is a thermo-couple, which measures the temperature. After the cessation of heating, the temperature at the point where the thermo-couple is situated gradually rises, reaches a maximum and then diminishes. The time τ_{max} corresponding to the maximum temperature t_{max}, and also the quantity of heat, Q_0, given out, are noted. These data are completely sufficient to determine the coefficient of thermal conductivity of that part of the body under investigation where the measuring instruments are placed. From special experiments carried out in advance, the curve $\lambda_q = f(u)$ is determined so that the moisture content of that portion of the body can be found from the coefficient of heat conductivity λ_q.

The coefficient of heat conductivity λ_q is determined from the formula

$$\lambda_q = \frac{Q_0}{4\pi\Delta t_{max}} \varphi_\lambda \tag{3.15}$$

where $\Delta t_{max} = t_{max} - t_0$. t_0 is the initial temperature, φ_λ is a certain coefficient depending on the ratio τ_0/τ_{max}. It is calculated by the formula

$$\varphi_\lambda = \int_{z_1}^{\infty} e^{-\theta} \frac{d\theta}{\theta} - \int_{z_2}^{\infty} e^{-\theta} \frac{d\theta}{\theta}, \tag{3.16}$$

$$z_1 = \frac{1 - \tau_0/\tau_{max}}{\tau_0/\tau_{max}} \ln \frac{1}{1 - \tau_0/\tau_{max}}; \quad z_2 = \frac{1}{\tau_0/\tau_{max}} \ln \frac{1}{1 - \tau_0/\tau_{max}},$$

or immediately from the graph of Fig. 44.

In addition wecan also determine the coefficient of thermal diffusivity a_q:

$$a_q = \frac{r_0^2}{4\tau_0}\varphi_a, \tag{3.17}$$

where $\varphi_a = \tau_0/z_1\tau_{\max}$.

The graph $\varphi_a = f(\tau_0/\tau_{\max})$ is drawn on the same Fig. 44.

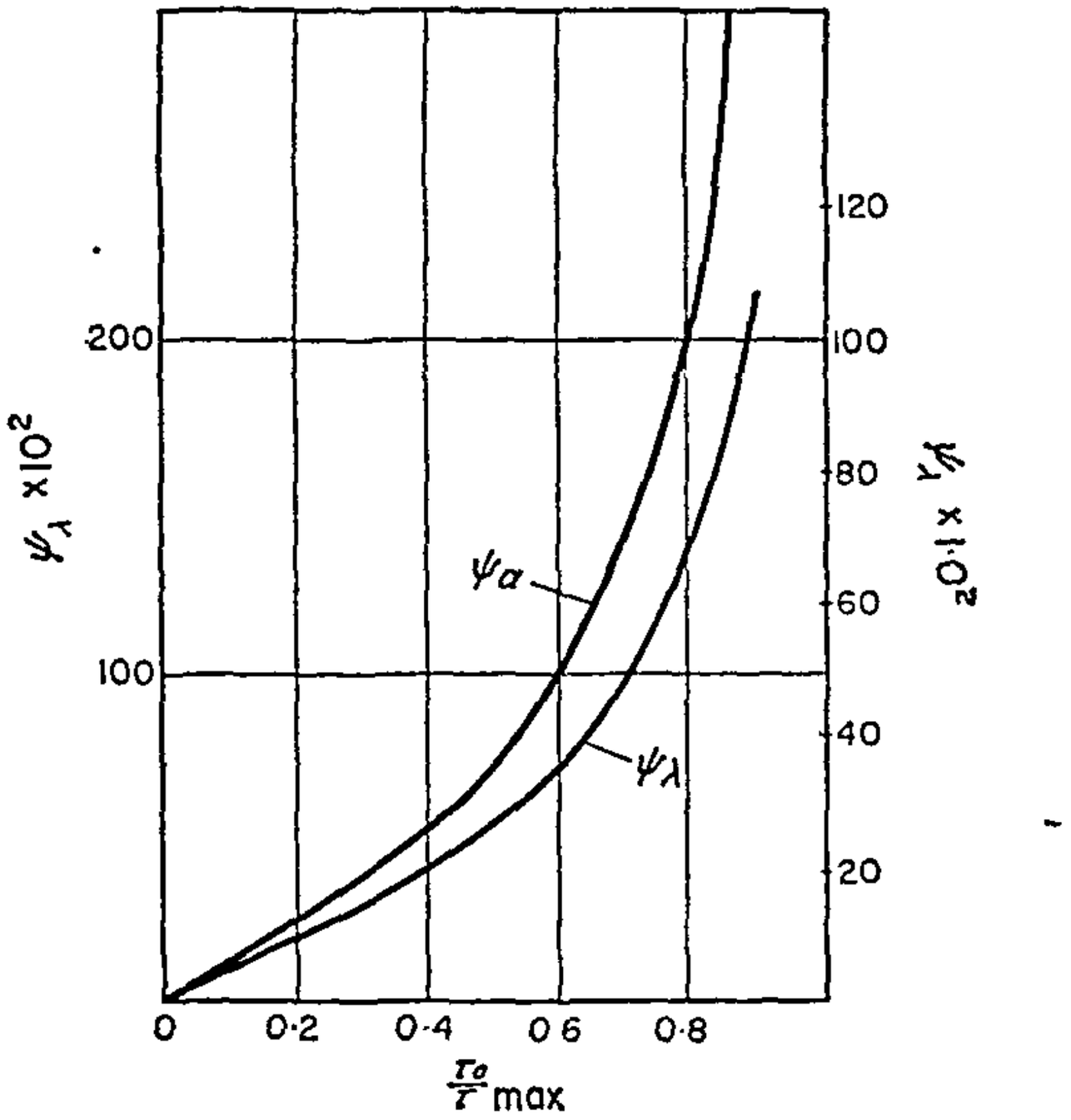


FIG. 44. Graph of the functions φ_a and φ_λ

Hence, the use of this instrument makes it possible to determine not only the coefficients of mass transfer (c_m, a_m, λ_m), but also the coefficients of heat transfer (c^q, a^q, λ^q).

The specific heat capacity is determined by the usual relation

$$c_q = \frac{\lambda_q}{a_q\gamma_0}. \tag{3.18}$$

The duration of the experiment is from 30 to 100 sec. Hence it is possible to carry out a series of repeated measurements and obtain average values. A disadvantage is the use of the auxiliary graphs $\lambda_q = f(u)$, for the construction of which the moisture content of the body is determined by the ordinary method of drying.

(d) *Bab'yev's Method* can be regarded as a universal method for granular materials. It enables us to determine, at the same time, the coefficients of transfer of heat and mass in the moist body. The essential part of the apparatus is a glass cylinder (150 mm long, 25 mm diameter), hung by its

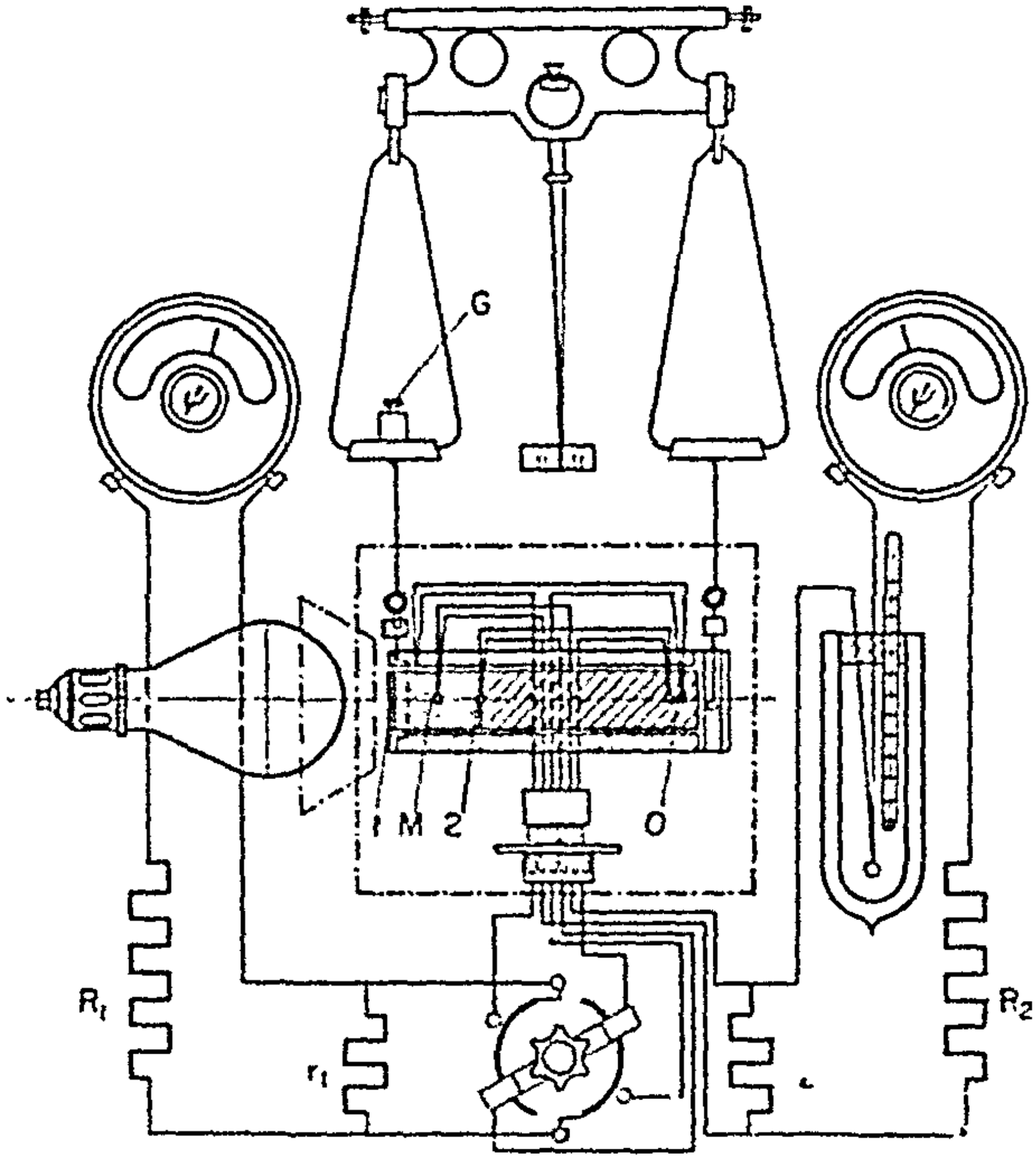


FIG. 45. Diagram of experimental layout

ends to the beam of an analytical balance (see Fig. 45). The distance between the points of suspension $L = 140$ mm is exactly equal to the distance between the knife edges of the beam. Part of the cylinder (45 mm long) is filled with paraffin (standard body) and the remainder is filled with the substance to be investigated. The paraffin insulates one end layer of the body under investigation against moisture. The second layer of the body (the end of the cylinder) is closed by a disc of copper foil 0·1 mm thick and 50 mm in

diameter. The copper disc ensures insulation against moisture. In addition it equalises the temperature over the cross-section of the cylinder and prevents the entry of infrared rays into the end, the cylinder being heated by radiation from a lamp. The temperature is measured at the following points: (1) In the plane of contact of the copper disc and the body under investigation t_1, (2) in the middle layer of the body under investigation t_M, (3) in the plane of contact of the body with the paraffin t_2, and (4) in the paraffin at a distance of 20 mm from the end of the cylinder t_0. All the junctions of the thermo-couples are on the axis of the cylinder. In order to protect the cylinder from the influence of currents of air it is placed in a thermostat. After the cylinder has been weighed, the lamp is switched on and, at the same time, the temperature is taken. From the values of t_1, t_2, t_0 the coefficients of heat transfer are determined. The formulae for calculation are obtained by the solution of the following problem of heat conduction.

A finite rod of length l (the body investigated) is in contact at one end with a semi-infinite rod (the paraffin), and the other end is heated by a constant flux of radiation of heat. There is no loss of heat from the side surfaces. From the statement of the problem it follows: (1) that the temperature at the distant point t_0 must not change during the experiment (imitation of a semi-infinite body), (2) the heat given out from the open end of the rod is small in comparison with the absorption of the flux of radiation.

The coefficient of thermal diffusivity a_q is determined from the formula

$$a_q = \frac{l^2}{4z^2\tau''}, \tag{3.19}$$

where τ'' is the time of the second measurement of temperature at point 1 for which the ratio $N_1''/N_1' = 2$, N_1 is the reading of the galvanometer at point 1. The coefficient z is determined from the graph of $z = f(\tau''/\tau')$ shown in Fig. 46.

The coefficient of heat conductivity is determined from the formula

$$\lambda_q = \lambda_2 \frac{1-h}{1+h} \sqrt{\left(\frac{a_q}{a_2}\right)}. \qquad (3.20)$$

where λ_2 and a_2 are respectively the coefficient of heat conductivity and the coefficient of thermal diffusivity of the

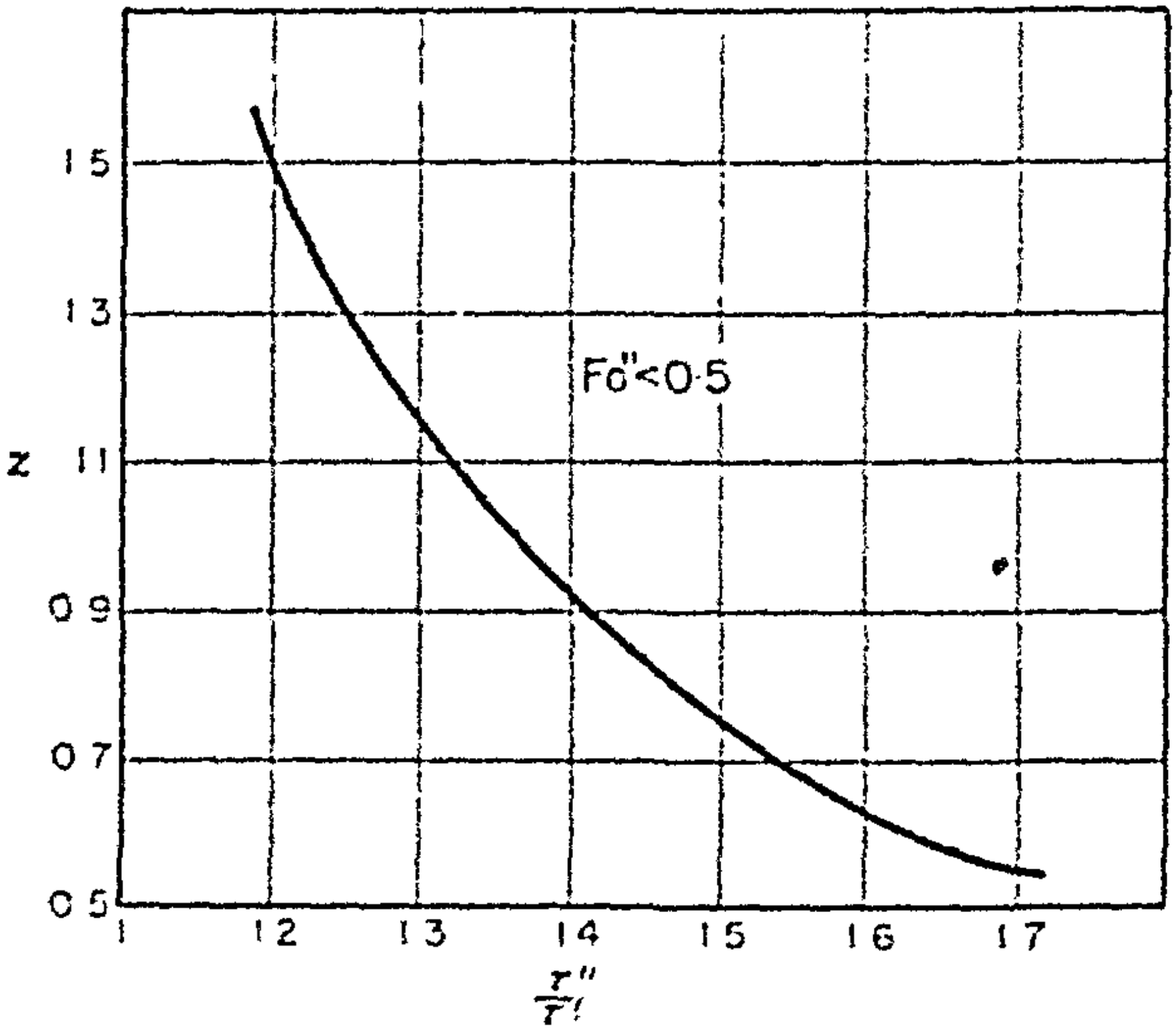


FIG. 46. Calculation of coefficient a_q

standard body (paraffin), h is an auxiliary quantity calculated from the formula

$$h = \frac{0.5642 \frac{N_1}{N_2} - i\,\mathrm{erfc}\,\frac{1}{2\sqrt{\mathrm{Fo}_1}}}{\frac{N_1}{N_2} i\,\mathrm{erfc}\,\frac{1}{\sqrt{\mathrm{Fo}_1}} - i\,\mathrm{erfc}\,\frac{1}{2\sqrt{\mathrm{Fo}_1}}}, \qquad (3.21)$$

where N_1/N_2 is the ratio of the readings of the galvanometer for thermocouples 1 and 2 in the time interval τ_1, $\mathrm{Fo}_1 = a_q \tau_1/l^2$ is the Fourier criterion for the same interval of time. The time required for all the readings is usually 10 — 15 min, including the reading of the analytical balance and the equilibration of temperature in the layer of the body. The cha-

racteristics of mass transfer a_m and δ are determined on the hypothesis of a linear law of distribution of moisture content through the length of the cylinder ($\mathrm{Fo}_m > 0{\cdot}2$)

$$a_m = \frac{q_m}{\gamma_0(\delta\nabla t + \nabla u)}. \tag{3.22}$$

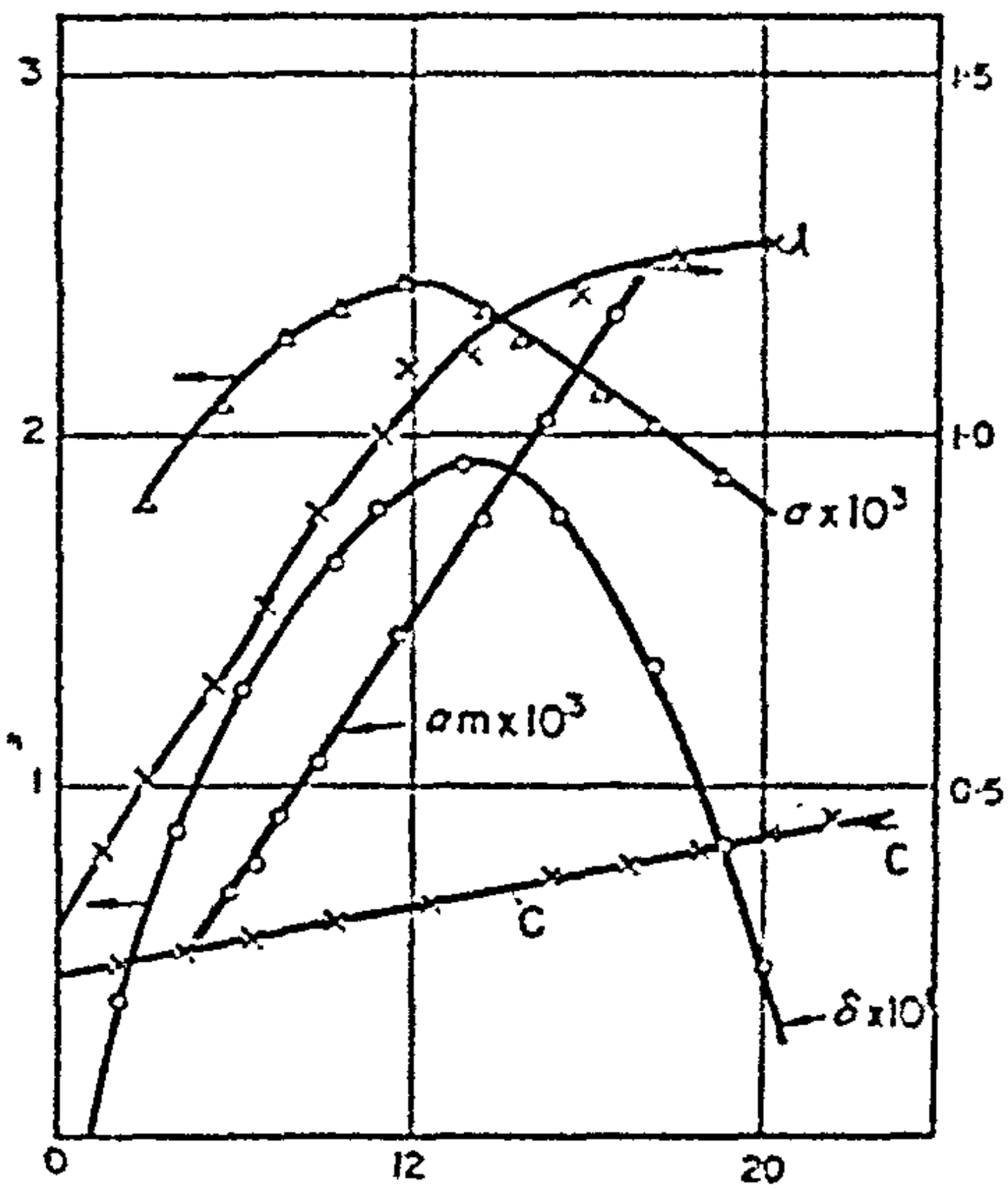


FIG. 47. Relation between the coefficients of heat and mass exchange for quartz sand

where q_m is the flux of moisture which passes from one end of the cylinder of the substance being investigated to the other. It is equal to

$$q_m = \frac{3L\,\Delta G}{2lA\,\Delta\tau}, \tag{3.23}$$

where ΔG is the weight required to re-establish the equilibrium of the balance after time $\Delta\tau$, and A is the area of the cross-section of the sample.

The gradient of moisture content is calculated from the formula

$$\nabla u = \frac{12\,GL}{G_0 l^2}, \tag{3.24}$$

where G is the weight required to re-establish the equilibrium of the balance, and G_0 is the weight of the dry substance investigated.

The thermo-gradient coefficient is calculated from the formula

$$\delta = \frac{12\,GL}{O_0 l^2 \nabla t}. \tag{3.25}$$

The coefficients of transfer of heat are determined from experimental data, the experiment lasting from 5 to 15 min. The mass exchange characteristics are determined from the appearance of the quasi-stationary condition ($\nabla t = \text{const}$ and $\nabla u = \text{const}$.), in 2 – 3 hr depending on the substance investigated.

The results of experiments with moist quartz sand, carried out by this method, are given in Fig. 47.

CHAPTER 4

Non-stationary Fields of Transfer Potential of Heat and Mass under Boundary Conditions of the Second Kind

In many cases the specific fluxes of heat and matter at the surface of a body cannot be described by the laws of convected heat and mass exchange. In these cases the specific fluxes of heat $q_q(\tau)$ and matter $q_m(\tau)$ are given as certain unknown functions of the time, to be determined by experiment.

In this case the boundary conditions for uni-dimensional bodies with symmetrical fields of transfer potential of heat and mass can be written thus:

$$-\lambda_y \frac{\partial t(R,\tau)}{\partial \xi} + q_q = 0\,, \tag{IX}$$

$$-\lambda_m \frac{\partial \theta(R,\tau)}{\partial \xi} + \lambda_m \delta \frac{\partial t(R,\tau)}{\partial \xi} + q_m = 0\,, \tag{X}$$

$$\frac{\partial t(0,\tau)}{\partial \xi} = 0, \quad \frac{\partial \theta(0,\tau)}{\partial \xi} = 0. \tag{XI}$$

It should be noted that the boundary conditions (IX – XI) reflect in their physical content a large class of heat exchange and mass exchange phenomena, including radiation heat exchange. Let us first consider problems where the coefficient δ is equal to 0. Solutions of such problems have been obtained by Smirnov for an infinite plate, cylinder and sphere, by the use of finite Fourier and Hankel transformations, i.e. the system of differential equations of transfer

$$\frac{\partial t(\xi,\tau)}{\partial\tau} = a_q\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial t(\xi,\tau)}{\partial\xi}\right] + \\ + \frac{\varepsilon_T \varrho}{c_q}\frac{\partial\vartheta(\xi,\tau)}{\partial\tau}, \tag{XII}$$

$$\frac{\partial\vartheta(\xi,\tau)}{\partial\tau} = a_m\left[\frac{\partial^2\vartheta(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\frac{\partial\vartheta(\xi,\tau)}{\partial\xi}\right], \tag{XIII}$$

$$t(\xi,0) = f(\xi),\ \vartheta(\xi,0) = \varphi(\xi), \tag{XIV}$$

$$\left.\begin{aligned}\frac{\partial t(0,\tau)}{\partial\xi} = 0,\ t(0,\tau) < \infty\\ \frac{\partial\vartheta(0,\tau)}{\partial\xi} = 0,\ \vartheta(0,\tau) < \infty\end{aligned}\right\}, \tag{XV}$$

$$-\lambda_q\frac{\partial t(R,\tau)}{\partial\xi} + q_q = 0, \tag{XVI}$$

$$\lambda_m\frac{\partial\vartheta(R,\tau)}{\partial\xi} + q_m = 0. \tag{XVII}$$

has been solved.

§ 1. Infinite Plate ($\delta = 0$)

$$[\Gamma = 0,\ \xi = x,\ \delta = 0,\ t(x,0) = f(x),\ \vartheta(x,0) = \varphi(x),$$

$$q_q = \text{const},\ q_m = \text{const}].$$

The solution of the system of differential equations (XII), (XIII) under boundary conditions (XVI), (XVII), an arbitrary initial distribution of transfer potentials (XIV), symmetry conditions (XV) and constant flux densities of heat and mass will be found by the method of finite integral Fourier transformation.

Let us determine first of all the distribution function of the transfer potential of mass in an infinite plate. For this purpose we solve equation (XIII).

The application of the integral transformation

$$\vartheta_{c.F}(n,\tau) = \int_0^R \vartheta(x,\tau)\cos\frac{n\pi x}{R}\,dx (n = 0,1,2,3\ldots) \tag{1.1}$$

with respect to the variable x to $\theta''(x, \tau)$ gives

$$\int_0^R \frac{\partial^2 \theta(x,\tau)}{\partial x} \cos \frac{n\pi x}{R} dx = (-1)^n \frac{\partial \theta(R,\tau)}{\partial x} -$$

$$- \frac{\partial \theta(0,\tau)}{\partial x} - \frac{n^2\pi^2}{R^2} \theta_{c.F}(n,\tau) ,$$

and, taking into account conditions (XV) and (XVII),

$$\int_0^R \frac{\partial^2 \theta(x,\tau)}{\partial x} \cos \frac{n\pi x}{R} dx$$

$$= -(-1)^n \frac{q_m}{\lambda_m} - \frac{n^2\pi^2}{R^2} \theta_{c.F}(n.\tau) . \qquad (1.2)$$

Further, multiplying the right hand and left hand side of the differential equation (XIII) by cos $n\pi x/R$, integrating within the limits 0 to R and taking into account the result (1.2), we obtain the ordinary differential equation with respect to τ:

$$\frac{d\,\theta_{c\cdot F}(n,\tau)}{d\tau} + \frac{a_m n^2\pi^2}{R^2} \theta_{c.F}(x,\tau) = -(-1)^n \frac{q_m}{\gamma_0 c'_T} .$$

The solution of this equation will be as follows:

$$\theta_{c.F}(n,\tau) = (-1)^n \frac{q_m}{\lambda_m} \frac{R^2}{n^2\pi^2} \left[\exp\left(-n^2\pi^2 \frac{a_m\tau}{R^2}\right) - 1\right] +$$

$$+ C_1(n) \exp\left(-n^2\pi^2 \frac{a_m\tau}{R^2}\right) . \qquad (1.3)$$

The constant of integration $C_1(n)$ can be found by using the initial condition $\theta(x, 0) = \varphi(x)$. From (1.3) we have, for $\tau \to 0$,

$$\theta_{c.F}(n, 0) = C_1(n) ,$$

but, on the other hand, by (1.1) and taking account of the initial distribution of transfer potentials of mass we have

$$\theta_{c.F}(n, 0) = \int_0^R \varphi(x) \cos \frac{n\pi x}{R} dx .$$

Consequently,

$$C_1(n) = \int_0^R \varphi(x) \cos \frac{n \pi x}{R} dx ,$$

and the solution for the representation can finally be written thus:

$$\theta_{c.F}(n, \tau) = (-1)^n \frac{q_m}{\lambda_m} \frac{R^2}{n^2 \pi^2} \left[\exp \left(- n^2 \pi^2 \frac{a_m \tau}{R^2} \right) - 1 \right] +$$

$$+ \exp \left(- n^2 \pi^2 \frac{a_m \tau}{R^2} \right) \int_0^R \varphi(x) \cos \frac{n \pi x}{R} dx . \qquad (1.4)$$

In order to find the distribution function of the transfer potential of mass it remains to carry out the transition from the transform $\theta_{c.F}(n, \tau)$ to its inversion $\theta(x, \tau)$ by means of the inversion formula for the Fourier cosine transformation:

$$\theta(x, \tau) = \frac{1}{R} \theta_{c.F}(0, \tau) + \frac{2}{R} \sum_{n=1}^{\infty} \theta_{c.F}(n, \tau) \cos \frac{n \pi x}{R} . \qquad (1.5)$$

For $n = 0$ the first term of (1.4) gives an indeterminacy of the form 0/0, which must be resolved by the rule of L'Hôpital. Carrying out the necessary transformations we obtain for the zero term of (1.5)

$$\frac{1}{R} \theta_{c.F}(0, \tau) = - \frac{q_m \tau}{c_T \gamma_0 R} + \frac{1}{R} \int_0^R \varphi(x) dx . \qquad (1.6)$$

Substituting from (1.4) and (1.6) in the expression (1.5) and remembering that, according to the theory of the Fourier series,

$$2R \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 \pi^2} \cos \frac{n \pi x}{R} = - \frac{R}{6} \left(1 - 3 \frac{x^2}{R^2} \right) ,$$

we have for the distribution $\theta(x, \tau)$ in an infinite plate

$$\theta(x, \tau) - \frac{1}{R} \int_0^R \varphi(x) dx = - \frac{q_m R}{\lambda_m} \left[\mathrm{Fo}^* - \frac{1}{6} \left(1 - 3 \frac{x^2}{R^2} \right) + \right.$$

(1.9) can be re-written in the following form

$$\frac{dt_{c.F}(n,\tau)}{d\tau} + \frac{a_q n^2 \pi^2}{R^2} t_{c.F}(n,\tau)$$

$$= (-1)^n \frac{a_q g_q}{\lambda_q} - \frac{\varepsilon_0 c'_T}{c_q}\left[(-1)^n \frac{g_m}{\gamma_0 c'_T} \exp\left(-n^2\pi^2 \frac{a_m \tau}{R^2}\right) + \right.$$

$$+ \frac{a_m n^2 \pi^2}{R^2} \exp\left(-n^2\pi^2 \frac{a_m \tau}{R^2}\right) \times$$

$$\left. \times \int_0^R \varphi(x) \cos\frac{x\pi x}{R} dx\right].$$

The solution of this ordinary non-homogeneous differential equation will be as follows:

$$t_{c.F} = (-1)^n \frac{g_q}{\lambda_q} \frac{R^2}{n^2\pi^2}[1 - \exp(-n^2\pi^2 \mathrm{Fo})] +$$

$$+ (-1)^n \frac{\varepsilon_0}{c_q} \frac{g_m}{\gamma_0} \frac{R^2}{n^2\pi^2(a_m - a_q)} \times$$

$$\times [\exp(-n^2\pi^2 \mathrm{Fo}^*) - \exp(-n^2\pi^2 \mathrm{Fo})] +$$

$$+ \frac{\varepsilon_0 c'_T}{c_q} \frac{a_m}{a_m - a_q}[\exp(-n^2\pi^2 \mathrm{Fo}^*) - \exp(-n^2\pi^2 \mathrm{Fo})] \times$$

$$\times \int_0^R \varphi(x) \cos\frac{n\pi x}{R} dx + C_2(n) \exp(-n^2\pi^2 \mathrm{Fo}). \quad (1.10)$$

If τ (and, consequently, Fo and Fo*) tends to 0, then from (1.10) it follows that $t_{c.F}(n, 0) = C_2(n)$. On the other hand.

$$t_{c.F}(n,0) = \int_0^R t(x,0) \cos\frac{n\pi x}{R} dx$$

$$= \int_0^R t(x) \cos\frac{n\pi x}{R} dx.$$

Consequently,

$$C_2(n) = \int_0^R f(x) \cos\frac{n\pi x}{R} dx.$$

Inverting by means of the formula

$$t(x, \tau) = \frac{1}{R} t_{c.F}(0; \tau) + \frac{2}{R} \sum_{n=1}^{\infty} t_{c.F}(n, \tau) \cos \frac{n \pi x}{R}$$

after a few transformations, analogous to those carried out in obtaining $\theta(x, \tau)$, the solution for $t(x, \tau)$ will be written in the form

$$t(x, \tau) - \frac{1}{R} \int_0^R f(x)\, dx = \frac{q_q R}{\lambda_q} \left[\mathrm{Fo} - \frac{1}{6} \left(1 - 3 \frac{x^2}{R^2} \right) + \right.$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2 \pi^2} \cos \frac{n \pi x}{R} \exp (- n^2 \pi^2 \mathrm{Fo}) +$$

$$+ \frac{2 \lambda_q}{R^2 q_q} \sum_{n=1}^{\infty} \exp (- n^2 \pi^2 \mathrm{Fo}) \cos \frac{n \pi x}{R} \times$$

$$\left. \times \int_0^R f(x) \cos \frac{n \pi x}{R} dx \right] + \frac{\varepsilon \varrho}{c_q} \frac{\mathrm{Lu}}{\mathrm{Lu} - 1} \frac{q_m R}{a_m \gamma_0} \times$$

$$\times \left\{ (\mathrm{Fo} - \mathrm{Fo}^*) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2 \pi^2} \cos \frac{n \pi x}{R} \times \right.$$

$$\times [\exp (- n^2 \pi^2 \mathrm{Fo}) - \exp (- n^2 \pi^2 \mathrm{Fo}^*)] - \frac{2 \lambda_m}{R^2 q_m} \times$$

$$\times \sum_{n=1}^{\infty} [\exp (- n^2 \pi^2 \mathrm{Fo}) - \exp (- n^2 \pi^2 \mathrm{Fo}^*)] \times$$

$$\left. \times \cos \frac{n \pi x}{R} \int_0^R \varphi(x) \cos \frac{n \pi x}{R} dx \right\}. \qquad (1.11)$$

In obtaining the inversion (1.11) we made use of the equality

$$\frac{\varepsilon \varrho}{c_q} \frac{q_m}{a_m \gamma_0} \frac{a_m \tau}{R}$$

$$= \frac{\varepsilon \varrho}{c_q} \frac{q_m}{a_m \gamma_0} \frac{a_m}{a_m - a_q} \left(\frac{a_m \tau}{R} - \frac{0_q \tau}{R} \right).$$

In the case where the initial distributions of the potential of transfer of heat and mass are constant throughout the plate, i.e.

$$t(x, 0) = t_0 = \text{const},$$

$$\theta(x, 0) = \theta_0 = \text{const},$$

from the solution of (1.7) and (1.11) as a particular case we obtain respectively

$$\Theta = \frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = \text{Ki}_m \left[\text{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2\pi^2} \cos\frac{n\pi x}{R} \exp(-n^2\pi^2 \text{Fo}^*) \right], \quad (1.12)$$

$$T = \frac{t(x, \tau) - t_0}{t_c - t_0} = Ki_q \left[\text{Fo} - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2\pi^2} \cos\frac{n\pi x}{R} \exp(-n^2\pi^2 \text{Fo}) \right] +$$

$$+ \frac{\varepsilon \text{Ko Ki}_m \text{Lu}}{\text{Lu} - 1} \left\{ \text{Fo} - \text{Fo}^* + \right.$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2\pi^2} \cos\frac{n\pi x}{R} \times$$

$$\left. \times [\exp(-n^2\pi^2 \text{Fo}) - \exp(-n^3\pi^2 \text{Fo}^*)] \right\}, \quad (1.13)$$

where

$$\text{Ki}_m = \frac{q_m R}{a_m \gamma_0 (u_0 - u_p)} = \frac{q_m R}{\lambda_m (\theta_0 - \theta_p)},$$

$$\text{Ki}_q = \frac{q_q R}{\lambda_q (t_c - t_0)} -$$

are the mass exchange and heat exchange criteria of Kirpichev.

§ 2. Infinite Cylinder ($\delta = 0$)

$$[\Gamma = 1,\ \xi = \tau,\ \delta = 0,\ t(\tau, 0) = f(\tau),\ \theta(\tau, 0) = \varphi(\tau),$$

$$q_q = \text{const.},\ q_m = \text{const.}].$$

We integrate the system of differential equations (XII – XIII) under the boundary conditions indicated in §1. This integration is performed by means of the finite Hankel integral transformation.

Let us find the distribution function of the transfer potential of mass $\theta(r, \tau)$ in the cylinder.

Applying the integral transformation

$$\theta_H(p_n, \tau) = \int_0^R r\,\theta(r, \tau)\,J_0(p_n r)\,dr\,, \qquad (2.1)$$

where p_n are the roots of the characteristic equation

$$J_0'(p_n R) = 0\,,$$

to the group of derivatives, and taking into account the boundary condition (XVII), we obtain

$$\int_0^R \left[\frac{\partial^2 \theta(r, \tau)}{\partial r^2} + \frac{1}{r}\,\frac{\partial \theta(r, \tau)}{\partial r}\right] r J_0(p_n r)\,dr$$

$$= -\frac{q_m}{\lambda_m} R J_0(p_n R) - p_n^2\,\theta_H(p_n \tau)\,. \qquad (2.2)$$

Multiplying both sides of equation (XIII) by $rJ_0(p_n, r)$, integrating with respect to r from 0 to R and taking into account (2.1), we obtain the following ordinary differential equation with respect to τ:

$$\frac{d\,\theta_H(p_n, \tau)}{d\tau} + a_m p_n^2\,\theta_H(p_n, \tau) = -\frac{q_m}{c_\tau' \gamma_0} R J_0(p_n R)\,.$$

The solution of this equation is

$$\theta_H(p_n, \tau) = C(p_n) \exp(-a_m p_n^2 \tau) -$$

$$-\frac{q_m R}{\lambda_m p_n^2} J_0(p_n R)\,[1 - \exp(-a_m p_n^2 \tau)]\,. \qquad (2.3)$$

The constant of integration $C(p_n)$ is found by making use of the initial condition for the potential of mass; we have

$$C(p_n) = \int_0^R \varphi(r)\,r J_0(p_n r)\,dr\,.$$

Hence expression (2.3.) for the distribution of the transform of the transfer potential of mass in the cylinder can be re-written thus:

$$\theta_H(p_n, \tau) = \exp(-a_m p_n^2 \tau) \int_0^R \varphi(r)\, r J_0(p_n r)\, dr -$$

$$- \frac{q_m R}{\lambda_m p_n^2} J_0(p_n R) [1 - \exp(-a_m p_n^2 \tau)] . \qquad (2.4)$$

The inversion of $\theta_H(p_n, \tau)$ can be found by the formula, which, in this case has the form

$$\theta(r, \tau) = \frac{2}{R^2} \theta_H(0, \tau) + \frac{2}{R^2} \sum_{n=1}^{\infty}{}' \theta_H(p_n, \tau) \frac{J_0(p_n r)}{J_0^2(p_n R)} .$$

Since

$$\theta_H(0, \tau) = - \frac{q_m R \tau}{c_T' \gamma_0} + \int_0^R \varphi(r)\, r dr$$

and

$$\sum_{n=1}^{\infty} \frac{2 J_0(p_n r)}{(p_n R)^2 J_0(p_n R)} = - \frac{1}{4} \left(1 - 2 \frac{r^2}{R^2}\right),$$

then for the distribution of the transfer potential of mass we obtain

$$\theta(r, \tau) - \frac{2}{R^2} \int_0^R \varphi(r)^r\, dr = - \frac{q_m R}{a_m \gamma_0 c_T'} \left[2\mathrm{Fo}^* - \frac{1}{4} \left(1 - 2 \frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2 J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}^*) \right] + \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \times$$

$$\times \exp(-\mu_n^2 \mathrm{Fo}^*) \int_0^R \varphi(r)\, J_0\left(\mu_n \frac{r}{R}\right) dr . \qquad (2.5)$$

In the solution (2.5) $\mu_n = p_n R$ are the roots of the characteristic equation

$$J_1(\mu) = 0 . \qquad (2.6)$$

The values of μ_n are shown in Table 37, p. 322.

The differential equation of the transform of the temperature of the cylinder can be obtained by applying to (XII) the finite integral Hankel transformation and taking into account the boundary condition (XVI). Carrying out the necessary operations we have

$$\frac{dt_H(p_n, \tau)}{d\tau} + a_c p_n^2 t_H(p_n, \tau) = \frac{a_c q_c R}{\lambda_c} J_0(p_n R) + \frac{\varepsilon_0 c_T}{c_c} \frac{d\theta_H(p_n, \tau)}{d\tau}, \tag{2.7}$$

where

$$t_H(p_n, \tau) = \int_0^R t(r, \tau)\, r J_0(p_n r)\, dr \text{ and } n = 0, 1, 2, \ldots$$

$\frac{d\theta_H(p_n, \tau)}{d\tau}$ will be found by differentiating (2.5) with respect to τ. We substitute the value of the derivative thus found in equation (2.7). As a result we obtain

$$\frac{dt_H(p_n, \tau)}{d\tau} + a_c p_n^2 t_H(p_n, \tau) = \frac{a_c q_c R}{\lambda_c} J_0(p_n R) - \frac{\varepsilon_0 c_T'}{c_c} \exp(-a_m p_n^2 \tau) \left\{ \frac{q_m R}{\gamma_0 c_T} J_0(p_n R) + a_m p_n^2 \times \right.$$

$$\left. \times \int_0^R \varphi(r)\, r J_0(p_n r)\, dr \right\}.$$

The solution of this differential equation, after making use of the corresponding initial conditions, for $a_m \neq a_c$, will be

$$t_H(p_n, \tau) = \frac{q_c R}{\lambda_c p_n^2} J_0(p_n R) [1 - \exp(-a_c p_n^2 \tau)] + \frac{1}{1} \exp(-a_c p_n^2 \tau) \int_0^R f(r)\, r J_0(p_n r)\, dr + \frac{1}{1} \frac{\varepsilon_0 c_T}{c_c} \frac{a_m}{a_m - a_c} \left\{ \frac{q_m R}{\chi_m \gamma_0 p_n^2 c_T'} J_0(p_n R) + \right.$$

$$\left. + \int_0^R \varphi(r)\, r J_0(p_n r)\, dr \right\} [\exp(-a_m p_n^2 \tau) - \exp(-a_c p_n^2 \tau)]. \tag{2.8}$$

The inversion of (2.8) is found by the relation

$$t(r,\tau) = \frac{2}{R^2} t_H(0,\tau) + \frac{2}{R^2} \sum_{n=1}^{\infty} t_H(p_n,\tau) \frac{J_0(p_n r)}{J_0^2(p_n R)} .$$

Since

$$t_H(0,\tau) = \frac{a_q q_q R\tau}{\lambda_q} - \frac{\varepsilon\varrho}{c_q} \frac{q_m R\tau}{\gamma_0} + \int_0^R f(r)\, r dr ,$$

the solution of the differential equation (XII), after a few transformations, is as follows:

$$t(r,\tau) - \frac{2}{R^2}\int_0^R f(r)\, r dr = \frac{q_q R}{\lambda_q}\left[2\mathrm{Fo} - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) - \right.$$

$$- \sum_{n=1}^{\infty} \frac{2J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}) +$$

$$+ \frac{\lambda_q}{q_q R^2} \sum_{n=1}^{\infty} \frac{2J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}) \times$$

$$\left. \times \int_0^R f(r)\, r J_0\left(\mu_n \frac{r}{R}\right) dr \right] + \frac{\varepsilon\varrho}{c_q} \frac{\mathrm{Lu}}{\mathrm{Lu} - 1} \frac{q_m R}{a_m \gamma_0} \times$$

$$\times \left\{ 2(\mathrm{Fo} - \mathrm{Fo}^*) - \sum_{n=1}^{\infty} \frac{2J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0^2(\mu_n)} [\exp(-\mu_n^2 \mathrm{Fo}) - \right.$$

$$- \exp(-\mu_n^2 \mathrm{Fo}^*)] - \frac{\lambda_m}{q_m R^2} \times$$

$$\times \sum_{n=1}^{\infty} \frac{2J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} [\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)] \times$$

$$\left. \times \int_0^R \varphi(r)\, r J_0\left(\mu_n \frac{r}{R}\right) dr \right\} . \qquad (2.9)$$

If the initial distribution of potentials of transfer in the cylinder is uniform and constant, i.e.

$$f(r) = t(r,0) = t_0 = \mathrm{const} \text{ and } \varphi(r) = \theta(r,0) = \theta_0 = \mathrm{const} ,$$

then, from the solutions (2.5) and (2.9), as a particular case, we obtain respectively

$$\Theta = \frac{\theta_0 - \theta(r,\tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[2\,\mathrm{Fo}^* - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp\left(-\mu_n^2\,\mathrm{Fo}^*\right) \right], \qquad (2.10)$$

$$T = \frac{t(r,t) - t_0}{t_c - t_0} = \mathrm{Ki}_q \left[2\,\mathrm{Fo} - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp\left(-\mu_n^2\,\mathrm{Fo}\right) \right] +$$

$$+ \frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m\,\mathrm{Lu}}{\mathrm{Lu} - 1} \left\{ 2\,(\mathrm{Fo} - \mathrm{Fo}^*) - \sum_{n=1}^{\infty} \frac{2J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \times \right.$$

$$\left. \times \left[\exp\left(-\mu_n^2\,\mathrm{Fo}\right) - \exp\left(-\mu_n^2\,\mathrm{Fo}^*\right)\right] \right\}. \qquad (2.11)$$

In obtaining (2.10) and (2.11) we made use of a property of the integrals

$$\int_0^R \varphi(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr \quad \text{and} \quad \int_0^R f(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr\,,$$

which for $p_n \neq 0$ ($\mu_n = p_n R$) are equal to 0.

§ 3. Sphere ($\delta = 0$)

$$[\Gamma = 2,\ \xi = r,\ \delta = 0,\ t(r,0) = f(r),\ \theta(r,0) = \varphi(r)\,,$$

$$q_q = \mathrm{const},\ q_m = \mathrm{const}]\,.$$

Let us now give the solution of the system of equations of internal heat and mass transfer for the sphere under the limiting conditions considered in the preceding paragraph. The system is integrated by the use of the finite integral transformation of Fourier.

Let us determine the distribution of the transfer potential of mass in the sphere. From equation (XIII), taking into account the conditions indicated above, it follows that

$$\int_0^R \left(\frac{\partial^2 \theta(r,\tau)}{\partial r^2} + \frac{2}{R} \frac{\partial \theta(r,\tau)}{\partial r} \right) r \frac{\sin p_n r}{p_n} dr$$

$$= - \frac{q_m R}{a_m \gamma c_T'} \frac{\sin p_n R}{p_n} - p_n^2 \theta_{s.F}(p_n, \tau),$$

where p_n is a root of the characteristic equation

$$\tan pR = pR, \tag{3.1}$$

and

$$\theta_{s.F} = \int_0^R r\, \theta(r,\tau) \frac{\sin p_n r}{p_n} dr.$$

Hence equation (XIII) is equivalent to the differential equation for the transform of θ:

$$\frac{d\theta_{s.F}(p_n,\tau)}{d\tau} + a_m p_n^2 \theta_{s.F}(p_n,\tau) = - \frac{q_m R}{\gamma_0 c_T'} \frac{\sin p_n R}{p_n}. \tag{3.2}$$

The solution of equation (3.2), taking into account the initial condition, will be

$$\theta_{s.F}(p_n,\tau) = \exp(-a_m p_n^2 \tau) \int_0^R \varphi(r)\, r \frac{\sin p_n r}{p_n} dr -$$

$$- \frac{q_m R^2}{a_m \gamma_0 c_T'} \frac{\sin p_n R}{p_n R} \frac{1}{p_n^2} [1 - \exp(-a_m p_n^2 \tau)]. \tag{3.3}$$

Applying to the last equation the inversion formula

$$\theta(r,\tau) = \frac{3}{R^3} \theta_{s.F}(0,\tau) +$$

$$+ \frac{2}{R} \sum_{n=1}^{\infty} \frac{p_n}{\sin^2 p_n R} \frac{\sin p_n r}{r} \theta_{s.F}(p_n,\tau),$$

we obtain the following expression for the transfer potential of mass

$$\theta(\tau,\tau)-\frac{3}{R^3}\int_0^R\varphi(r)\,r^2\,dr$$

$$=-\frac{q_m R}{c_T'\,a_m\gamma_0}\left[\frac{3a_m\tau}{R^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{2}{p_n R\sin p_n R}\,\frac{\sin p_n r}{p_n r}-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2}{p_n R\sin p_n R}\,\frac{\sin p_n r}{p_n r}\exp\left(-a_m p_n^2\tau\right)\right]+$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}\frac{2}{R\sin^2 p_n R}\,\frac{p_n\sin p_n r}{r}\exp\left(-a_m p_n^2\tau\right)\times$$

$$\times\int_0^R\varphi(r)\,r\,\frac{\sin p_n r}{p_n}\,dr\,.$$

Taking into account equation (3.1), and also remembering that

$$\sum_{n=1}^{\infty}\frac{2}{p_n R\sin p_n R}\,\frac{\sin p_n r}{p_n r}=-\frac{1}{10}\left(3-5\,\frac{r^2}{R^2}\right),$$

the solution can finally be rewritten thus:

$$\theta(\tau,\tau)-\frac{3}{R^3}\int_0^R\varphi(r)\,r^2\,dr$$

$$=-\frac{q_m R}{a_m\gamma_0 c_T'}\left[3\,\mathrm{Fo}^*-\frac{1}{10}\left(3-5\,\frac{r^2}{R^2}\right)-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\dfrac{r}{R}}{\mu_n\dfrac{r}{R}}\exp\left(-\mu_n^2\,\mathrm{Fo}^*\right)\right]+$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}\frac{2}{\mu_n\cos^2\mu_n}\,\frac{\sin\mu_n\dfrac{r}{R}}{Rr}\exp\left(-\mu_n^2\,\mathrm{Fo}^*\right)\times$$

$$\times\int_0^R\varphi(r)\,r\,\frac{\sin\mu_n\dfrac{r}{R}}{\mu_n}\,dr\,. \qquad (3.4)$$

Here $\mu_n = p_n R$ are the roots of the equation (3.1). Their values are set out in Table 34.

Let us now find the distribution in the sphere of the transfer potential of heat. The differential equation for the transform of temperature will be

$$\frac{dt_{s.F}(p_n, \tau)}{d\tau} a_q p_n^2 t_{s.F}(p_n, \tau)$$

$$= \frac{a_q q_q R}{\lambda_q} \frac{\sin p_n R}{p_n} + \frac{\varepsilon_Q c'_T}{c_q} \frac{d\theta_{s.F}(p_n, \tau)}{d\tau}, \qquad (3.5)$$

where

$$t_{s.F}(p_n \tau) = \int_0^R t(r, \tau) r \frac{\sin p_n r}{p_n} dr,$$

and p_n is a root of the characteristic equation (3.1).

The derivative $d\theta_{s.F}(p_n, \tau)/d\tau$ will be determined from equation (3.3); substituting the value found in (3.5) we have

$$\frac{dt_{s.F}(p_n, \tau)}{d\tau} + a_q p_n^2 t_{s.F}(p_n, \tau) = \frac{a_q q_q R^2}{\lambda_q} \frac{\sin p_n R}{p_n R} -$$

$$- \exp(-a_m p_n^2 \tau) \frac{\varepsilon_Q c'_T}{c_q} \left\{ \frac{q_m R^2}{\gamma_0 c'_T} \frac{\sin p_n R}{p_n R} + \right.$$

$$\left. + a_m p_n^2 \int_0^R \varphi(r) r \frac{\sin p_n r}{p_n} dr \right\}.$$

The solution of this equation is

$$t_{s.F}(p_n, \tau) = \frac{q_q R^2}{\lambda_q} \frac{\sin p_n R}{p_n R} \frac{1}{p_n^2} [1 - \exp(-a_q p_n^2 \tau)] +$$

$$+ \exp(-a_q p_n^2 \tau) \int_0^R f(r) r \frac{\sin p_n r}{p_n} dr +$$

$$+ \frac{\varepsilon_Q c'_T}{c_q} \frac{a_m}{a_m - a_q} \left\{ \frac{q_m R^2 \sin p_n R}{a_m \gamma_0 p_n^2 c'_T p_n R} + \right.$$

$$\left. + [\exp(-a_m p_n^2 \tau) - \exp(-a_q p_n^2 \tau)] \int_0^R \varphi(r) r \frac{\sin p_n r}{p_n} dr \right\}.$$

In view of the fact that the inversion formula for the temperature has the form

$$t(r, \tau) = \frac{3}{R^2} t_{s.F}(0, \tau) + \frac{2}{R} \sum_{n=1}^{\infty} \frac{p_n}{\sin^2 p_n R} \frac{\sin p_n r}{r} t_{s.F}(p_n, \tau),$$

the final solution for the distribution of temperature in the sphere can be written, after transformations, in the following form:

$$t(r, \tau) - \frac{3}{R^3} \int_0^R f(r)\, r^2\, dr = \frac{q_q R}{\lambda_q} \left[3\mathrm{Fo} - \frac{1}{10} \left(3 - 5 \frac{r^2}{R^2} \right) - \right.$$

$$- \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp(-\mu_n^2 \mathrm{Fo}) +$$

$$+ \frac{\lambda}{q_q R} \sum_{n=1}^{\infty} \frac{2}{\mu_n \cos^2 \mu_n} \frac{\sin \mu_n \frac{r}{R}}{Rr} \exp(-\mu_n^2 \mathrm{Fo}) \times$$

$$\left. \times \int_0^R f(r)\, r \frac{\sin \mu_n \frac{r}{R}}{\mu_n} dr \right] + \frac{\varepsilon \varrho}{c_q} \frac{\mathrm{Lu}}{\mathrm{Lu} - 1} \frac{q_m R}{a_m \gamma_0} \left\{ 3 (\mathrm{Fo} - \mathrm{Fo}^*) - \right.$$

$$- \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} [\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)] -$$

$$- \frac{a_m \gamma_0 c'_T}{q_m R} \sum_{n=1}^{\infty} \frac{2}{\mu_n \cos^2 \mu_n} \frac{\sin \mu_n \frac{r}{R}}{Rr} \times$$

$$\times [\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)] \times$$

$$\left. \times \int_0^R \varphi(r)\, r \frac{\sin \mu_n \frac{r}{R}}{\mu_n} dr \right\}. \qquad (3.6)$$

If $\varphi(r) = \theta_0 = \text{const.}$, and $f(r) = t_0 = \text{const.}$, then the solutions for the transfer potentials of mass and heat (3.4) and

(3.6) are transformed respectively into solutions (3.7) and (3.8):

$$\Theta = \frac{\theta_0 - \theta(r,\tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[3\,\mathrm{Fo}^* - \frac{1}{10}\left(3 - 5\frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left(-\mu_n^2 \mathrm{Fo}^*\right) \right], \qquad (3.7)$$

$$T = \frac{t(r,\tau) - t_0}{t_c - t_0} = \mathrm{Ki}_q \left[3\,\mathrm{Fo} - \frac{1}{10}\left(3 - 5\frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left(-\mu_n^2 \mathrm{Fo}\right) \right] +$$

$$+ \frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m\,\mathrm{Lu}}{\mathrm{Lu} - 1} \left\{ 3\,(\mathrm{Fo} - \mathrm{Fo}^*) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \left[\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)\right] \right\}. \qquad (3.8)$$

§ 4. Analysis of the Solutions

Let us analyse the analytical solutions obtained in §§ 1 – 3 (equations (1.7), (1.11)–(1.13), (2.5), (2.9)–(2.11), (3.4), (3.6)–(3.8). In the case of a constant and uniform initial distribution of transfer potentials, the distribution of temperature in the body is produced by the superposition of three temperature fields. The first field, described by the expression in the square bracket, characterizes the heating of the material without taking into account mass exchange or phase transformation in the body. The second field represents the influence of stationary sources or sinks of heat (the terms in Fo in the second group of terms). Finally, the third field represents the influence of moving sources of heat, i.e. phase changes in the moving matter (terms in Fo*, in the second group of terms).

In the case where the criterion $\varepsilon = 0$, i.e. when there are no phase-changes in the material, all three solutions are transformed into the ordinary classical solutions of the equation of heat conduction without sources of heat.

Everything that has been said is true also for the more general solutions, obtained on the supposition that the initial distribution of temperature and potential of mass is not uniform or constant, but is given in the form of a function of the co-ordinates. In this case the solutions contain additional terms representing the influence of the initial conditions.

The series which enter into the solutions converge rapidly, so that the solutions obtained are convenient for engineering calculations.

The distribution of the transfer potential of mass in the body corresponds to the classical solutions of the equations of diffusion and conduction. As the time, or more precisely the criterion Fo* increases, the terms of the series contained in the solutions rapidly diminish and, for a certain value of $\mathrm{Fo}^* > \mathrm{Fo}_1^*$, become negligibly small in comparison with the two first terms, so that the whole series may be neglected. Beginning from this moment of time, $\theta(\xi, \tau)$ at any point of the body will be a linear function of the time, and its distribution in the body will obey a parabolic law, i.e. the quasi-stationary condition will be established.

§ 5. Infinite Cylinder $(\delta \neq 0)$

$$[\Gamma = 1, \quad \xi = r, \quad t(r, 0) = f_1(r), \quad \theta(r, 0) = f_2(r),$$
$$q_q = q_q(\tau), \quad q_m = q_m(\tau)] .$$

Let us solve the system of equations (I′ – II′) under boundary conditions (IX – XI) and initial conditions (XIV) by the method of the simultaneous application of the integral transformations of Hankel and Laplace. In distinction from the preceding paragraphs, in order to shorten the writing out of further transformations, we will denote the transfer potentials of heat and mass by θ_i $(i = 1, 2)$, so that $\theta_1 = t$ and

$\theta_2 = \theta$, and, in addition, the subscripts q and m in the coefficients will be respectively replaced by 1 and 2.

We shall make use of the finite integral transformation of Hankel with respect to the variable r :

$$\{\theta_i(p, \tau)\}_H = \int_0^R r\,\theta_i(r, \tau)\, J_0(pr)\, dr\,, \qquad (5.1)$$

where p is the root of the characteristic equation $J_0'(pR) = 0$.

The transition from the transform $\{\theta_i(p, \tau)\}_H$ to the inversion $\theta_i(r, \tau)$ is carried out by means of the formula

$$\theta_i(r, \tau) = \frac{2}{R^2} \int_0^R r\,\theta_i(r, \tau)\, dr + \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{J_0(p_n r)}{J_0^2(p_n R)} \{\theta_i(p, \tau)\}_H. \qquad (5.2)$$

The application of the Hankel transformation (5.1), taking into account the boundary conditions (IX – XI), to the group of terms of equations (I – II) which stand in the square brackets, leads to the following relations

$$\int_0^R \left[\frac{\partial^2 \theta_1(r, \tau)}{\partial r^2} + \frac{1}{r}\,\frac{\partial \theta_1(r, \tau)}{\partial r}\right] r J_0(pr)\, dr$$

$$= R\frac{q_1(\tau)}{\lambda_1} J_0(pR) - p^2 \{\theta_1(p, \tau)\}_H\,, \qquad (5.3)$$

$$\int_0^R \left[\frac{\partial^2 \theta_2(r, \tau)}{\partial r^2} + \frac{1}{r}\,\frac{\partial^2 \theta_2(r, \tau)}{\partial r}\right] r J_0(pr)\, dr$$

$$= -R\left[\delta\frac{q_1(\tau)}{\lambda_1} + \frac{q_2(\tau)}{\lambda_2}\right] J_0(pR) - p^2 \{\theta_2(p, \tau)\}_H\,. \qquad (5.4)$$

Assuming that the operator of the Hankel transformation $\{\}_H$ is commutative with the operator of differentiation, $\partial/\partial\tau$, after multiplying all the members of the equation (I – II) by $rJ_0(pr)$, and integrating with respect to r from 0 to R taking account of (5.3)–(5.4), the initial system of differential equations in partial derivatives can be transformed into a system of two ordinary differential equations:

$$\left.\begin{aligned}
\frac{d}{d\tau}\{\theta_1(p,\tau)\}_H &= a_1\left[R\frac{q_1(\tau)}{\lambda_1}J_0(pR) - \right.\\
&\left. - p^2\{\theta_1(p,\tau)\}_H\right] + \varepsilon_0\frac{c'}{c}\frac{d}{d\tau}\{\theta_2(p,\tau)\}_H\\
\frac{d}{d\tau}\{\theta_2(p,\tau)\}_H &= -a_2\left[R\frac{q_2(\tau)}{\lambda_2}J_0(pR) + \right.\\
&\left. + p^2\{\theta_2(p,\tau)\}_H\right] - a_2\,\delta\,p^2\{\theta_1(p,\tau)\}_H
\end{aligned}\right\}. \qquad (5.5)$$

In the derivation of equation (5.5) it was assumed that the conditions (XIV), after the application of the Hankel transformation (5.1), were transformed into the conditions

$$\{\theta_i(p,0)\}_H = \{f_i(p)\}_H \quad (i=1,2)\,. \qquad (5.6)$$

Let us now apply to the system of differential equations (5.5) and the initial conditions (5.6) the Laplace integral transformation

$$\{\theta_i(\tau,s)\}_L = s\int_0^\infty \theta_i(\tau,\tau)\,e^{-s\tau}\,d\tau\,. \qquad (5.7)$$

As a result we obtain the following algebraic system of equations:

$$(s + a_1 p^2)\{\theta_i(p,s)\}_{HL} - \frac{\varepsilon_0 c'}{c}s\{\theta_2(p,s)\}_{HL}$$

$$= s\{f_1(p)\}_H + \frac{a_1}{\lambda_1}RJ_0(pR)\{q_1(s)\}_L - \frac{\varepsilon_0 c'}{c}s\{f_2(p)\}_H\,,$$

$$a_2\,\delta\,p^2\{\theta_1(p,s)\}_{HL} + (s + a_2 p^2)\{\theta_2(p,s)\}_{HL}$$

$$= s\{f_2(p)\}_H - \frac{a_2}{\lambda_2}RJ_0(pR)\{q_2(s)\}_L\,.$$

Solving the last named system of equations for $\{\theta_i(p,s)\}_{HL}$ $(i=1,2)$ and obtaining the inversion of these potentials with respect to the parameter s, we obtain

$$\{\theta_i(p,s)\}_H = \sum_{j,k=1}^{2}\left[A^i_{jk}\{P_{jk}\}_H + B^i_{jk}\{Q_{jk}\}_H\right], \qquad (5.8)$$

where

$$A^1_{j1} = (-1)^j \frac{1 - v_j^2}{v_1^2 - v_2^2}, \quad A^1_{j2} = (-1)^j \frac{\varepsilon_0 \frac{c'}{c}}{v_2^2 - v_1^2},$$

$$B^1_{j1} = (-1)^j \frac{a_1}{\lambda_1} \frac{1 - v_j^2}{v_1^2 - v_2^2},$$

$$B^1_{j2} = (-1)^j v_j^2 \frac{a_2}{\lambda_2} \frac{\varepsilon_0 \frac{c'}{c}}{v_1^2 - v_2^2} = (-1)^j \frac{a_2}{\lambda_2 \delta} - \frac{\varepsilon \mathrm{Ko\,Pn}\, v_3^2}{v_1^2 - v_2^2},$$

$$A^2_{j1} = (-1)^j \frac{\delta}{v_2^2 - v_1^2}, \quad A^2_{j2} = (-1)^j \left[\frac{\frac{1}{\mathrm{Lu}} - v_j^2 + \varepsilon \mathrm{Ko\,Pn}}{v_1^2 - v_2^2} \right],$$

$$B^2_{j1} = (-1)^j \frac{a_2}{\lambda_1} \frac{\delta}{v_2^2 - v_1^2}, \quad B^2_{j2} = (-1)^j \frac{a_2}{\lambda_2} \frac{\frac{1}{\mathrm{Lu}} - v_j^2}{v_1^2 - v_2^2},$$

$$v_j^2 = \frac{1}{2} \left\{ \left(1 + \varepsilon \mathrm{Ko\,Pn} + \frac{1}{\mathrm{Lu}} \right) + (-1)^{j+1} \sqrt{\left[\left(1 + \varepsilon \mathrm{Ko\,Pn} + \frac{1}{\mathrm{Lu}} \right)^2 - \frac{4}{\mathrm{Lu}} \right]} \right\},$$

$$\{P_{jk}\}_H = \{f_k(p)\}_H \exp\left(-R^2 p^2 v_j^2 \mathrm{Fo}_2\right),$$

$$\{Q_{jk}\}_H = R J_0(pR) \int_0^\tau g_k(\eta) \times$$

$$\times \exp\left[-R^2 p^2 v_j^2 \frac{a_2(\tau - \eta)}{R^2} \right] d\eta,$$

$$\mathrm{Fo}_2 = \frac{a_2 \tau}{R^2} = \mathrm{Lu\,Fo}, \quad (j = 1,2).$$

It is not difficult to prove that the function $\{\theta_i(p, \tau)\}_H$, defined by the expression (5.8), satisfies the initial conditions (5.6).

Let us now find the inversion of the transform $\{\theta_i(p, \tau)\}_H$, using for this purpose formula (5.2). After a few transformations we obtain

$$\theta_i(r, \tau) = \sum_{j,k=1}^{2} \left(A^i_{jk} P_{jk} + B^i_{jk} Q_{jk} \right) \quad (i = 1,2),$$

where

$$P_{jk} = \frac{2}{R^2}\left[\int_0^R r f_k(r)\, dr + \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \times \right.$$

$$\left. \times \exp\left(-\mu_n^2 v_j^2 \mathrm{Fo}_2\right) \int_0^R r f_k(r)\, J_0\left(\mu_n \frac{r}{R}\right) dr \right],$$

$$Q_{jk} = \frac{2}{R}\left[\int_0^\tau q_k(\eta)\, d\eta + \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0(\mu_n)} \times \right.$$

$$\left. \times \exp\left(-\mu_n^2 v_j^2 \mathrm{Fo}_2\right) \int_0^R q_k(\eta) \exp\left(\mu_n^2 v_j^2 \frac{a_2 \eta}{R^2}\right) \right] d\eta\,,$$

and μ_n are the roots of the equation $J_0'(\mu) = 0$.

§ 6. Sphere $(\delta \neq 0)$

$$[\Gamma = 2,\ \xi = r,\ t(r, 0) = f_1(r),\ \theta(r, 0) = f_2(r),$$

$$q_q = q_q(\tau), q_m = q_m(\tau)]\,.$$

Let us solve this problem under the same conditions as in the preceding paragraph. We will use the method of simultaneous application of the integral sine-transformations of Fourier and the Laplace transformation. The sine-transformation of Fourier mentioned already in §3 has the form

$$\{\theta_i(p, \tau)\}_F = \int_0^R r\, \theta_i(r, \tau) \frac{\sin pr}{p}\, dr\,, \qquad (6.1)$$

where p is a root of a characteristic equation

$$\tan pR = pR\,.$$

By analogy with §§ 3 and 5, applying first of all the transformation (6.1) and then the Laplace transformation (5.7) to the system of equations (I – II) and the boundary conditions

(IX – XI), (XIV), we obtain the algebraical system of equations

$$(s + a_1 p^2)\{\theta_1(p, s)\}_{FL} - \frac{\varepsilon_0 c'}{c} s\{\theta_2(p, s)\}_{FL}$$

$$= s\{f_1(p)\}_F + R\frac{a_1}{\lambda_1}\frac{\sin pR}{p}\{q_1(s)\}_L - \frac{\varepsilon_0 c'}{c} s\{f_2(p)\}_F, \quad (6.2)$$

$$a_2 \delta p^2 \{\theta_1(p, s)\}_{FL} + (s + a_2 p^2)\{\theta_2(p, s)\}_{FL}$$

$$= s\{f_2(p)\}_F - R\frac{a_2}{\lambda_2}\frac{\sin pR}{p}\{q_2(s)\}_L. \quad (6.3)$$

Solving the system of equations (6.2) and (6.3) for $\{\theta_i(p, s)\}_{FL}$ $(i = 1, 2)$ and passing to the inversions with respect to s, we obtain

$$\{\theta_i(p, \tau)\}_F = \sum_{j,k=1}^{2} [A'_{jk}\{P_{jk}\}_F + B'_{jk}\{Q_{jk}\}_F], \quad (6.4)$$

where A'_{jk} and B'_{jk} are defined by the expressions introduced in § 5 and

$$\{P_{jk}\}_F = \{f_k(p)\}_F \exp[-R^2 p^2 \nu_j^2 \mathrm{Fo}_2],$$

$$\{Q_{jk}\}_F = \frac{R\sin pR}{p}\int_0^\tau q_k(\eta)\exp\left[p^2 R^2 \nu_j^2 \frac{a_2(\tau - \eta)}{R^2}\right] d\eta.$$

The inverse Fourier transformation for the problem under consideration may be obtained from the formula

$$\theta_i(r, \tau) = \frac{3}{R^3}\int_0^R r^2 f_i(r, \tau)\, dr +$$

$$+ \frac{2}{R}\sum_{n=1}^{\infty}\frac{\mu_n}{\sin^2 \mu_n R}\frac{\sin \mu_n r}{r}\{\theta_i(p, \tau)\}_F. \quad (6.5)$$

Applying formula (6.5) to the equality (6.4), we obtain, or the transfer potential of heat and mass, finally,

$$\theta_i(r, \tau) = \sum_{j,k=1}^{2}(A'_{jk} P_{jk} + B'_{jk} Q_{jk}),$$

where

$$P_{jk} = \frac{3}{R^3}\int_0^R r^2 f_k(r)\,dr + \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\mu_n}{\sin^2\mu_n}\times$$

$$\times\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}\exp\left(-\mu_n^2\nu_j^2\,\mathrm{Fo}_2\right)\int_0^R r f_k(r)\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n}\,dr\,,$$

$$Q_{jk} = \frac{3}{R^2}\int_0^{\tau} q_k(\eta)\,d\eta + \frac{2}{R}\sum_{n=1}^{\infty}\frac{\sin\mu_n\frac{R}{R}}{\frac{r}{R}\sin\mu_n}\times$$

$$\times\exp\left(-\mu_n^2\nu_j^2\,\mathrm{Fo}_2\right)\int_0^{\tau} q_k(\eta)\exp\left(\mu_n^2\nu_j^2\frac{a_2\eta}{R^2}\right)d\eta\,,$$

and μ_n are the roots of the equation

$$\tan\mu = \mu\,.$$

§ 7. Infinite Plate. Generalized Boundary Conditions of the Second Kind

Let us solve the system of equations (I′ – II′) under the initial conditions (XIV) and the symmetry conditions (XV), but under boundary conditions having the form

$$\frac{\partial\theta_1(R,\tau)}{\partial x} + \alpha\theta_1(R,\tau) + \beta\theta_2(R,\tau) = \varphi_1(\tau)\,, \qquad (7.1)$$

$$\frac{\partial\theta_2(R,\tau)}{\partial x} + \alpha'\frac{\partial\theta_1(R,\tau)}{\partial x} + \beta'\,\theta_2(R,\tau) = \varphi_2(\tau)\,. \qquad (7.2)$$

Here, as in the preceding paragraphs, $\theta_1 = t$, $\theta_2 = \theta$, and α, β, α', β' are defined by the aggregate of known thermophysical coefficients.

Following Prudnikov, let us consider the auxiliary problem in which the boundary conditions (7.1) and (7.2) are replaced by the conditions

$$\theta_k(R,\tau) = \chi_k(\tau)\,, \qquad (k = 1, 2)\,. \quad (7.3)$$

This will be solved by the method of the simultaneous application of the integral transformations of Fourier and Laplace.

With respect to the variable x we apply the finite Fourier cosine-transformation:

$$\{\theta_i(p,\tau)\}_F = \int_0^R \theta_i(x,\tau)\cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x\,dx\,. \qquad (7.4)$$

The original of the transform of the transfer potentials $\{\theta_i(p,\tau)\}_F$ is determined by the formula

$$\theta_i(x,\tau) = \frac{2}{R}\sum_{n=1}^{\infty}\{\theta_i(p,\tau)\}_F\cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x\,. \qquad (7.5)$$

With respect to the variable τ we apply the Laplace transformation

$$\{\theta_i(x,s)\}_L = \int_0^{\infty} e^{-s\tau}\,\theta(x,\tau)\,d\tau\,. \qquad (7.6)$$

By means of the successive application of the integral transformations (7.4), (7.6), taking into account the initial conditions (XIV) and the boundary conditions (7.3), the original problem is reduced to an algebraical system of equations, and applying to these the inverse Laplace transformation we obtain.

$$\{\theta_i(p,\tau)\}_F = \sum_{j,s=1}^{2}\left[A^i_{js}\{v_{js}\}_F + B^i_{js}\{w_{js}\}_F\right]\ (i=1,2), \qquad (7.7)$$

where

$$\{v_{js}\}_F = \exp\left[-\left(n+\frac{1}{2}\right)^2\pi^2\nu_j^2\,\mathrm{Fo}_2\right]\{f_s(p)\}_F\,,$$

$$\{w_{js}\}_F = (-1)^n\left(n+\frac{1}{2}\right)\frac{\pi}{R}\times$$

$$\times\exp\left[-\left(n+\frac{1}{2}\right)^2\pi^2\nu_j^2\,\mathrm{Fo}_2\right]^{\tau}*\chi_s(\tau)\,,$$

$$\nu_j^2 = \frac{1}{2}\left\{\left(1+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}+\frac{1}{\mathrm{Lu}}\right)+(-1)^{j+1}\times\right.$$

$$\left.\times\sqrt{\left[\left(1+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}+\frac{1}{\mathrm{Lu}}\right)^2-\frac{4}{\mathrm{Lu}}\right]}\right\},$$

$$\mathrm{Fo}_2 = \frac{a_2\tau}{R^2}\,.$$

The symbol, $\overset{\tau}{*}$ denotes the operation of convolution:

$$f(\tau)\overset{\tau}{*} g(\tau) = \int_0^\tau f(\eta)\, g(\tau-\eta)\, d\eta = \int_0^\tau f(\tau-\eta)\, g(\eta)\, d\eta = g(\tau)\overset{\tau}{*} f(\tau).$$

The constant coefficients A^i_{js}, B^i_{js} $(i, j, s = 1, 2)$ are determined by the formulae:

$$A^1_{j1} = (-1)^j \frac{1 - \nu_j^2}{\nu_1^2 - \nu_2^2},\quad A^1_{j2} = (-1)^j \frac{\varepsilon\varrho \dfrac{c'}{c}}{\nu_2^2 - \nu_1^2},$$

$$B^1_{j1} = (-1)^j \frac{a_1 [1 - \nu_j^2 - \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu}\,\nu_j^2]}{\nu_1^2 - \nu_2^2},$$

$$B^1_{j2} = (-1)^j \frac{a_2}{\delta}\,\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\frac{\nu_j^2}{\nu_2^2 - \nu_1^2},$$

$$A^2_{j1} = (-1)^j \frac{\delta}{\nu_2^2 - \nu_1^2},$$

$$A^2_{j2} = (-1)^j \frac{\nu_j^2 - \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} - \dfrac{1}{\mathrm{Lu}}}{\nu_2^2 - \nu_1^2},$$

$$B^2_{j2} = (-1)^j a^2 \delta \frac{\nu_j^2}{\nu_2^2 - \nu_1^2},$$

$$B^2_{j2} = (-1)^j \frac{a_2 \left(\nu_j^2 - \dfrac{1}{\mathrm{Lu}}\right)}{\nu_2^2 - \nu_1^2}.$$

The application of the inversion formula (7.5) to the expression (7.7) gives

$$\theta_i(x, \tau) = \sum_{j,s=1}^{2} (A^i_{js}\, v_{js} + B^i_{js}\, w_{js})\ (i = 1, 2), \qquad (7.8)$$

where

$$v_{js} = \frac{2}{R} \sum_{n=0}^{\infty} \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \nu_j^2\, \mathrm{Fo}_2\right] \times$$

$$\times \int_0^R f_s(\xi) \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R}\,\xi\, d\xi \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R}\, x, \qquad (7.9)$$

$$w_{js} = \frac{2}{R} \sum_{n=0}^{\infty} \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \nu_j^2\, \mathrm{Fo}_2\right]\overset{\tau}{*}(-1)^n \times$$

$$\times \left(n + \frac{1}{2}\right)\frac{\pi}{R}\,\chi_s(\tau) \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R}\, x. \qquad (7.10)$$

The expressions (7.9) and (7.10) can be simplified by introducing the theta-function: after carrying out the necessary transformation we obtain

$$v_{js} = \frac{1}{2R}\int_0^R \left\{\vartheta_2\left[\frac{x-\xi}{2R}, \nu_j^2 \mathrm{Fo}_2\right] + \vartheta_2\left[\frac{x+\xi}{2R}, \nu_j^2 \mathrm{Fo}_2\right]\right\} f_s(\xi)\, d\xi\,,$$

$$v_{js} = \frac{1}{R}\int_0^\tau \frac{\partial}{\partial x}\vartheta_2\left[\frac{R-x}{2R}, \nu_j^2 \frac{a_2(\tau-\eta)}{R_2}\right] \chi_s(\eta)\, d\eta\,, \qquad (7.11)$$

$$\vartheta_2(x,\tau) = \frac{1}{\sqrt{\pi\tau}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{(x+n)^2}{\tau}\right]. \qquad (7.12)$$

In order to solve the problem of this paragraph it is necessary to determine the unknown functions $\chi_k(\tau)$ from the boundary conditions (7.1) and (7.2). Differentiating (7.8) with respect to x, and replacing the second derivative of ϑ_2 with respect to x by its derivative with respect to τ, we obtain

$$\frac{\partial\theta_j(x,\tau)}{\partial x} = \sum_{j,s=1}^{2} (A_{js}^j P_{js} + B_{js}^j Q_{js}), \quad (i = 1,2)\,, \qquad (7.13)$$

where

$$P_{js}(x,\tau) = \frac{1}{2R}\int_0^R \frac{\partial}{\partial x}\left\{\vartheta_2\left[\frac{x-\xi}{2R}, \nu_j^2 \mathrm{Fo}_2\right] + \right.$$

$$\left. + \vartheta_2\left[\frac{x+\xi}{2R}, \nu_j^2 \mathrm{Fo}_2\right]\right\} f_s(\xi)\, d\xi\,,$$

$$Q_{js}(x\tau) = \frac{1}{\nu_j^2 a_2 R}\int_0^\tau \frac{\partial}{\partial(\tau-\eta)}\vartheta_2\left[\frac{R-x}{2R}, \nu_j^2 \times\right.$$

$$\left. \times \frac{a_2(\tau-\eta)}{R^2}\right] \chi_s(\eta)\, d\eta\,. \qquad (7.14)$$

It is not difficult to see that the derivative $\partial\theta_j/\partial(\tau-\eta)$, which exists for all values of $x \neq R$, tends to infinity like $\tau^{-3/2}$, as $\tau \to 0$ when $x = R$ and $\tau \to \eta$; hence we will replace the boundary conditions (7.1) and (7.2) by the condi-

tions which are obtained from the latter by integration with respect to τ. In this case we obtain:

$$\int_0^\tau \frac{\partial \theta_1(R,\eta)}{\partial x}\, d\eta + a \int_0^\tau \chi_1(\eta)\, d\eta + \beta \int_0^\tau \chi_2(\eta)\, d\eta$$

$$= \int_0^\tau \varphi_1(\eta)\, d\eta\,, \qquad (7.15)$$

$$\int_0^\tau \frac{\partial \theta_2(R,\eta)}{\partial x}\, d\eta + a' \int_0^\tau \frac{\partial \theta_1(R,\eta)}{dx}\, d\eta + \beta' \int_0^\tau \chi_2(\eta)\, d\eta$$

$$= \int_0^\tau \varphi_2(\eta)\, d\eta\,. \qquad (7.16)$$

Integrating (7.13) with respect to τ we find

$$\int_0 \frac{\partial \theta_l(x,\eta)}{\partial x}\, d\eta = \sum_{j,s=1} (A^l_{js} P^*_{js} + B^l_{js} Q^*_{js}) \ (=1,2)\,, \quad (7.17)$$

where

$$P^*_{js}(x,\tau) = \int_0^\tau P_{js}(x,\eta)\, d\eta\,,\ Q^*_{js}(x,\tau) = \int_0^\tau Q_{js}(x,\eta)\, d\eta\,.$$

Integrating (7.14) with respect to τ and changing the order of integration by means of the Dirichlet formula, we obtain

$$Q^*_{js}(x,T) = \frac{1}{v_j^2 a_2 R} \int_0^T d\tau \int_0^\tau \chi_s(\eta) \frac{\partial}{\partial(\tau-\eta)} \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R},\ v_j^2 \frac{a^2(\tau-\eta)}{R^2}\right] d\eta$$

$$= \frac{1}{v_j^2 a_2 R} \int_0^T \chi_s(\eta)\, d\eta \int_\eta^T \frac{\partial}{\partial(\tau-\eta)} \vartheta_2 \left[\frac{R-x}{2R}, v_j^2 \times\right.$$

$$\left. \times \frac{a_2(\tau-\eta)}{R^2}\right] d\tau = \frac{1}{v_j^2 a_2 R} \times$$

$$\times \int_0^T \chi_s(\eta)\, d\eta \left\{\vartheta^2 \left[\frac{R-x}{2R}, v_j^2 \frac{a_2(\tau-\eta)}{R^2}\right] - \vartheta_2 \left[\frac{R-x}{2R}, 0\right]\right\}.$$

Hence it follows that, for all values of x less than R,

$$Q_{js}^{*}(x,T) = \frac{1}{\nu_j^2 a_2 R}\int_0^T \chi_s(\eta)\,\vartheta_2\left[\frac{R-x}{2R}, \nu_j^2 \times\right.$$

$$\left.\times \frac{a_2(T-\eta)}{R^2}\right] d\eta\,. \qquad (7.18)$$

Calculations show that (7.18) holds also for x equals R, i.e.

$$Q_{js}^{*}(R,T) = \frac{1}{\nu_j^2 a_2 R}\int_0^T \chi_s(\eta)\,\vartheta^2\left[0, \nu_j^2\frac{a_2(T-\eta)}{R^2}\right] d\eta\,. \quad (7.19)$$

The last integral converges, since by (7.12) the function ϑ_2 for $\eta \to T$ tends to infinity as $T \to 0$. Making use of (7.17) — (7.19) to obtain the corresponding terms under the modified boundary conditions (7.15) and (7.16), and also remembering (7.3), we obtain a system of integral Volterra equations of the first kind:

$$\sum_{p=1}^{2}\int_0^\tau K_{\alpha\beta}(\tau-\eta)\,\chi_\beta(\eta)\,d\eta = g_\alpha(\tau)\ \ (\alpha = 1,2)\,, \qquad (7.20)$$

where

$$K_{11}(\tau-\eta) = \frac{B_{11}^1}{\nu_1^2 a_2 R}\vartheta_2\left[0, \nu_1^2\,\frac{a_2(\tau-\eta)}{R^2}\right] +$$

$$+\frac{B_{21}^1}{\nu_2^2 a_2 R}\vartheta_2\left[0, \nu_2^2\,\frac{a_2(\tau-\eta)}{R^2}\right] + \alpha\,,$$

$$K_{12}(\tau-\eta) = \frac{B_{12}^1}{\nu_1^2 a_2 R}\vartheta_2\left[0, \nu_1^2\frac{a_2(\tau-\eta)}{R^2}\right] +$$

$$+\frac{B_{22}^1}{\nu_2^2 a_2 R}\vartheta_2\left[0, \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right] + \beta\,,$$

$$K_{21}(\tau-\eta) = \frac{\alpha' B_{11}^1 + B_{11}^2}{\nu_1^2 a_2 R}\vartheta_2\left[0, \nu_1^2\,\frac{a_2(\tau-\eta)}{R^2}\right] +$$

$$+\frac{\alpha' B_{21}^1 + B_{21}^2}{\nu_2^2 a_2 R}\vartheta_2\left[0, \nu_2^2\,\frac{a_2(\tau-\eta)}{R^2}\right]\,,$$

$$K_{22}(\tau-\eta) = \frac{\alpha' B_{12}^1 + B_{12}^2}{\nu_1^2 a_2 R}\vartheta_2\left[0, \nu_1^2\,\frac{a_2(\tau-\eta)}{R^2}\right] +$$

$$+\frac{\alpha' B_{22}^1 + B_{22}^2}{R^2}\vartheta_2\left[0, \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right] + \beta',$$

$$g_1(\tau) = \int_0^\tau \varphi_1(\eta)\, d\eta - \sum_{j,s=1}^{2} A_{js}^1 P_{js}^*(R,\tau)\,,$$

$$g_2(\tau) = \int_0^\tau \varphi_2(\eta)\, d\eta - \sum_{j,s=1}^{2} (\alpha' A_{js}^1 + A_{js}^2) P_{js}^*(R,\tau)\,.$$

Let us write

$$K_{11}(\tau-\eta) = \frac{G_1(\tau-\eta)}{\sqrt{(\tau-\eta)}},\ K_{22}(\tau-\eta) = \frac{G_2(\tau-\eta)}{\sqrt{(\tau-\eta)}}, \quad (7.21)$$

where

$$G_1(\tau-\eta) = \frac{\varkappa_1}{\sqrt{\pi}} + \frac{B_{11}^1}{\nu_1^2 a_2 \sqrt{(\nu_1^2 a_2 \pi)}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2 n^2}{\nu_1^2 a_2(\tau-\eta)}\right] + \frac{B_{21}^1}{\nu_2^2 a_2 \sqrt{(\nu_2^2 a_2 \pi)}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2 n^2}{\nu_2^2 a_2(\tau-\eta)}\right] + \alpha\sqrt{(\tau-\eta)}\,,$$

$$G_2(\tau-\eta) = \frac{\varkappa_2}{\sqrt{\pi}} + \frac{\alpha' B_{22}^1 + B_{12}^2}{\nu_1^2 a_2 \sqrt{(\nu_1^2 a_2 \pi)}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2 n^2}{\nu_1^2 a_2(\tau-\eta)} + \frac{\alpha' B_{22}^1 + B_{22}^2}{\nu_2^2 a_2 \sqrt{(\nu_2^2 a_2 \pi)}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times \right.$$

$$\times \exp\left[-\frac{R^2 n^2}{\nu_1^2 a_2(\tau-\eta)}\right] + \beta'\sqrt{(\tau-\eta)}\,,$$

$$\varkappa_1 = \frac{B_{11}^1}{\nu_1^1 a_2 \sqrt{(\nu_1^2 a_2)}} + \frac{B_{21}^1}{\nu_2^2 a_2 \sqrt{(\nu_2^2 a_2)}}\,,$$

$$\varkappa_2 = \frac{\alpha' B_{12}^1 + B_{12}^2}{\nu_1^2 a \sqrt{(\nu_1^2 a_2)}} + \frac{\alpha' B_{22}^1 + B_{22}^2}{\nu_2^2 a_2 \sqrt{(\nu_2^2 a_2)}}\,.$$

The summation Σ' in these expressions extends to all integral values of n except $n = 0$. Substituting the expressions (7.21) in (7.20), we obtain a system of generalized integral equations

of Abel, which can be reduced to a system of generalized Volterra equations of the first kind:

$$\varphi_\alpha(\tau) + \sum_{\beta=1}^{2} \int_0^\tau \varphi_{\alpha\beta}(\tau, \eta)\, \varphi_\beta(\eta)\, a\eta = \omega_\alpha(\tau) \quad (\alpha = 1, 2), \quad (7.22)$$

where

$$\varphi_{\alpha\beta}(\tau, \eta) = \frac{1}{\varkappa_\alpha \sqrt{\pi}} \frac{d}{d\tau} L_{\alpha\beta}(\tau, \eta),$$

$$\omega_\alpha(\tau) = \frac{1}{\varkappa_\alpha \sqrt{\pi}} \frac{d}{d\tau} s_\alpha(\tau),$$

$$L_{\alpha\beta}(\tau, \eta) = \int_\eta^\tau \frac{C_\alpha(T-\eta)}{\sqrt{[(\tau - T)(T-\eta)]}}\, dT \qquad (\alpha = \beta),$$

$$L_{\alpha\beta}(\tau, \eta) = \int_\eta^\tau \frac{K_{\alpha\beta}(T-\eta)}{\sqrt{(\tau - T)}}\, dT \qquad (\alpha \neq \beta),$$

$$s_\alpha(\tau) = \int_0^\tau \frac{g_\alpha(T)}{\sqrt{(\tau - T)}}\, dT,$$

and α and β assume the values 1, 2.

The final solution of the problem will be obtained after determining $\varphi_\alpha(\tau)$ from the system of equations (7.22) and substituting in equations (7.8) and (7.11).

§ 8. Certain Special Problems with Generalized Boundary Conditions

We give here solutions of the system of equations (I – II) for a symmetrical infinite plate and a sphere with constant initial conditions and with boundary conditions which form a special case of (7.1) and (7.2), and, to a certain degree also, of conditions (III) and (IV):

$$-\lambda_q t'(R, \tau) + \alpha [t_a - t(R, \tau)] - (1 - \varepsilon)\, \varrho\, q_m = 0, \quad (8.1)$$

$$\lambda_m \theta'(R, \tau) + \lambda_m \delta t'(R, \tau) + q_m = 0, \quad (8.2)$$

where $q_m = \text{const}$.

(a) *Infinite Plate*

The solution of the system of equations (I – II), in transform, with boundary conditions (8.1)–(8.2) has the form:

$$t_L(x, s) = \frac{t_0}{s} + \sum_{i=1}^{2} B_i \cosh \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, x\,, \qquad (8.3)$$

$$\theta_L(x, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon \varrho} \sum_{i=1}^{2} B_i (1 - \nu_i^2) \cosh \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, x\,, \qquad (8.4)$$

where

$$B_1 = -(t_a - t_0) \frac{\varepsilon \,\mathrm{Ko}\,\mathrm{Ki}_m\, Q_2 - \left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\dfrac{\mathrm{Ki}_m}{\mathrm{Bi}_q}\right] P_2}{S(Q_2 P_1 - Q_1 P_2)}\,,$$

$$B_2 = (t_a - t_0) \frac{\varepsilon \,\mathrm{Ko}\,\mathrm{Ki}_m\, Q_1 - \left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\dfrac{\mathrm{Ki}_m}{\mathrm{Bi}_q} - 1\right] P_1}{S(Q_2 P_1 - Q_1 P_2)}\,,$$

$$Q_i = \cosh \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, R + \frac{1}{\mathrm{Bi}_q} \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, R \sinh \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, R\,,$$

$$P_i = [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_i^2)] \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, R \sinh \sqrt{\left(\frac{s}{a_q}\right) \nu_i}\, R\,,$$

and ν_i is determined by the expression (1.5) Chapter 2.

In order to pass to the inversion we will find the roots of the denominator B_i:

$$\psi(s) = s(Q_2 P_1 - Q_1 Q_2)$$

$$= s \left\{ [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_1^2)] \sqrt{\left(\frac{s}{a_q}\right) \nu_1}\, R \sinh \sqrt{\left(\frac{s}{a_q}\right) \nu_1}\, R \cdot Q_2 - \right.$$

$$\left. - [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_2^2)] \left[\sqrt{\left(\frac{s}{a_q}\right) \nu_2}\, R \sinh \sqrt{\left(\frac{s}{a_q}\right) \nu_2}\, R \cdot Q_1 \right] \right\}$$

$$= s^2 \left\{ [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_1^2)] \left[\frac{(\nu_1 R)^2}{a_q} + \frac{(\nu_1 R)^4}{3!\, a_q^2} s + \ldots \right] Q_2 - \right.$$

$$\left. - [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_1^2)] \left[\frac{(\nu_2 R)^2}{a_q} + \frac{(\nu_2 R)^4}{3!\, a_q^2} s + \ldots \right] Q_1 \right\}$$

$$= S^2 \varphi(s) = 0\,, \qquad (8.5)$$

where $\varphi(s)$ is the expression in the compound brackets.

From the expression (8.5) we have:

(1) a twofold zero root, $s = s_0 = 0$;

(2) $s = s_m$, the roots of the joint solution of the system of equations:

$$\begin{cases} \sin \nu_1 \mu = 0 \\ \sin \nu_2 \mu = 0 \end{cases} \tag{8.6}$$

where $s_m = -\frac{\mu_m^2 a_q}{R^2}$;

(3) $s = s_n$ are the roots of the equation

$$\varkappa \cot \nu_1 \mu - \cot \nu_2 \mu + \frac{\nu_2 - \nu_1}{\mathrm{Bi}} \mu = 0, \tag{8.7}$$

where

$$S_n = -\frac{a_q \mu_n^2}{R^2},$$

$$\varkappa = \frac{\varepsilon \mathrm{Ko\,Pn} + (1 - \nu_2^2)}{\varepsilon \mathrm{Ko\,Pn} + (1 - \nu_1^2)} \cdot \frac{\nu_2}{\nu_1}.$$

The inversion of solution (8.3)–(8.4) we will find from the generalized expansion theorem for multiple roots. For a non-stationary distribution of the transfer potentials of heat and mass, we then obtain:

$$T(X, \mathrm{Fo}) = \frac{t - t_0}{t_c - t_0} = 1 - (1 - \varepsilon)\,\mathrm{Ko\,Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q} -$$

$$- \frac{1}{2}\varepsilon\,\mathrm{Ko\,Lu\,Ki}_m \left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) -$$

$$- \sum_{n=1}^{\infty} \sum_{l=1}^{2} C_{nl} \cos \nu_l \mu_n x/R \exp(-\mu_n^2 \mathrm{Fo}) +$$

$$+ 2\frac{\varepsilon\,\mathrm{Ko\,Ki}_m}{\nu_2^2 - \nu_1^2} \sum_{m=1}^{\infty} \frac{1}{\mu_m^2} \left(\frac{\cos \nu_2 \mu_m x/R}{\cos \nu_2 \mu_m} - \right.$$

$$\left. - \frac{\cos \nu_1 \mu_m x/R}{\cos \nu_1 \mu_m}\right) \exp(-\mu_m^2 \mathrm{Fo}), \tag{8.8}$$

$$\Theta(X, \mathrm{Fo}) = \frac{\theta_0 - \theta}{\theta_0 - \theta_c} = \mathrm{Ki}_m \left[\mathrm{Lu\,Fo} - (1 + \varepsilon\,\mathrm{Ko\,Pn\,Lu})\frac{1 - 3\frac{x^2}{R^2}}{6}\right] +$$

$$+\frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{n=1}^{\infty}\sum_{i=1}^{2}C_{ni}(1-\nu_i^2)\cos\nu_i\mu_n x/R\exp(-\mu_n^2\,\mathrm{Fo})+$$

$$+\frac{\mathrm{Ki}_m}{\nu_2^2-\nu_1^2}\sum_{m=1}^{\infty}\frac{1}{\mu_m^2}\left[(1-\nu_1^2)\frac{\cos\nu_1\mu_m x/R}{\cos\nu_1\mu_m}-\right.$$

$$\left.-(1-\nu_2^2)\frac{\cos\nu_2\mu_m x/R}{\cos\nu_2\mu_m}\right]\exp(-\mu_m^2\,\mathrm{Fo}), \tag{8.9}$$

where

$$C_{n1}=\quad 2\,\frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m Q_{n2}-\left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\dfrac{\mathrm{Ki}_m}{\mathrm{Bi}_q}-1\right]\mathrm{Pn}_2}{\mu_n\psi_n}, \tag{8.10}$$

$$C_{n2}=-2\,\frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m Q_{n1}-\left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\dfrac{\mathrm{Ki}_m}{\mathrm{Bi}_q}-1\right]\mathrm{Pn}_1}{\mu_n\psi_n}, \tag{8.11}$$

$$\psi_n=\nu_1 A_{n1}P_{n2}+\nu_2 B_{n2}Q_{n1}-\nu_2 A_{n2}P_{n1}-\nu_1 B_{n1}Q_{n2} \tag{8.12}$$

$$Q_{ni}=\cos\nu_i\mu_n-\frac{1}{\mathrm{Bi}_q}\nu_i\mu_n\sin\nu_i\mu_n,$$

$$P_{ni}=-\left[\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}+(1-\nu_i^2)\right]\nu_i\mu_n\sin\nu_i\mu_n,$$

$$A_{ni}=\left(1+\frac{1}{\mathrm{Bi}_q}\right)\sin\nu_i\mu_n+\frac{1}{\mathrm{Bi}_q}\nu_i\mu_n\cos\nu_i\mu_n,$$

$$B_{ni}=\sin\nu_i\mu_n+\nu_i\mu_n\cos\nu_i\mu_n.$$

The characteristic roots μ_n are determined from the solutions of equation (8.7), and μ_m from the solutions of the system of equations (8.6).

Since the infinite series converges very quickly, for a quasi-regular state we have

$$T(X,\mathrm{Fo})=1-(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q}-$$

$$-\frac{1}{2}\varepsilon\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Ki}_m\left(1-\frac{x^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right),$$

$$\Theta(X,\mathrm{Fo})=\mathrm{Ki}_m\left[\mathrm{Lu}\,\mathrm{Fo}-(1+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu})\,\frac{1-3\,x^2/R^2}{6}\right].$$

Consequently, in this special case, we obtain the solution of Polonskaya shown in Chapter 2 § (5)c.

(b) *Sphere.* The solution of the system of equations (I – II), in transform under boundary conditions (8.1)–(8.2) can be written in the following form:

$$t_L(r, s) = \frac{t_0}{s} + \sum_{l=1}^{2} B_l \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_l r}{r}, \qquad (8.13)$$

$$\theta_L(r, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon Q} \sum_{l=1}^{2} B_l (1 - \nu_l^2) \frac{\sinh \sqrt{\left(\frac{s}{a_q}\right)} r_l r}{r}, \quad (8.14)$$

where

$$B_1 = -(t_c - t_0) R \cdot \frac{\varepsilon \,\mathrm{Ko}\,\mathrm{Ki}_m Q_2 - \left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q} - 1\right] P_2}{s(Q_2 P_1 - Q_1 P_2)},$$

$$B_2 = (t_c - t_0) R \cdot \frac{\varepsilon \,\mathrm{Ko}\,\mathrm{Ki}_m Q_1 - \left[(1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q} - 1\right] P_1}{s(Q_2 P_1 - Q_1 P_2)},$$

$$Q_l = \left(1 - \frac{1}{\mathrm{Bi}_q}\right) \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R + \frac{1}{\mathrm{Bi}_q} \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R \cosh \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R,$$

$$P_l = [\varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + (1 - \nu_l^2)] \left[\sqrt{\left(\frac{s}{a_q}\right)} \nu_l R \cosh \sqrt{\left(\frac{s}{a_q}\right)} \times \right.$$

$$\left. \times \nu_l R - \sinh \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R \right].$$

As in the case of an infinite plate, it can be shown that the roots of the denominator B_l are:

(1) $s = s_0 = 0$ are double zero roots,

(2) $s = s_n$ are the roots of the expression

$$\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2 = 0,$$

where

$$\tilde{Q}_l = \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R \cdot Q_l \text{ and } \tilde{P}_l = \sqrt{\left(\frac{s}{a_q}\right)} \nu_l R \cdot P_l.$$

After applying to the equations (8.13) and (8.14) the generalized expansion theorem for multiple roots, we obtain a

solution of the problem under consideration in the following form:

$$T(X, \mathrm{Fo}) = 1 - (1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q} - \frac{1}{2}\,\varepsilon\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Ki}_m \times$$

$$\times\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) + \sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}\,\frac{\sin \nu_i\,\mu_n\,x/R}{x/R}\exp(-\mu_n^2\,\mathrm{Fo}), \tag{8.15}$$

$$\Theta(X, \mathrm{Fo}) = \mathrm{Ki}_m\left[3\,\mathrm{Lu}\,\mathrm{Fo} - (1 + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu})\,\frac{3 - 5\,x^2/R^2}{10}\right] -$$

$$-\frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}(1-\nu_i^2)\,\frac{\sin \nu_i\,\mu_n\,x/R}{x/R}\exp(-\mu_n^2\,\mathrm{Fo}). \tag{8.16}$$

Here

$$P_{ni} = [\varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + (1-\nu_i^2)]\,(\nu_i\,\mu_n\cos\nu_i\,\mu_n - \sin\nu_i\,\mu_n),$$

$$Q_{ni} = \left(1 - \frac{1}{\mathrm{Bi}_q}\right)\sin\nu_i\,\mu_n + \frac{1}{\mathrm{Bi}_q}\,\nu_i\,\mu_n\cos\nu_i\,\mu_n,$$

$$A_{ni} = \left(1 - \frac{1}{\mathrm{Bi}_q}\right)\left(\cos\nu_i\,\mu_n - \frac{\sin\nu_i\,\mu_n}{\nu_i\,\mu_n}\right) - \frac{1}{\mathrm{Bi}_q}\,\nu_i\,\mu_n\sin\nu_i\,\mu_n,$$

$$B_{ni} = [\varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + (1-\nu_i^2)]\left(\frac{\sin\nu_i\,\mu_n}{\nu_i\,\mu_n}\right) - \nu_i\,\mu_n\sin\nu_i\,\mu_n - \cos\nu_i\,\mu_n.$$

C_{ni} and ψ_n are determined respectively from the expressions (8.10), (8.11) and (8.12).

The characteristic roots μ_n are determined from the solutions of the equation

$$\tilde{Q}_{n2}\,\tilde{P}_{n1} - \tilde{Q}_{n1}\,\tilde{P}_{n2} = 0$$

where

$$\tilde{Q}_{ni} = \frac{1}{\nu_i\,\mu_n}\,Q_{ni},\quad \tilde{P}_{ni} = \frac{1}{\nu_i\,\mu_n}\,P_{ni}.$$

In the quasi-regular state the distribution of the transfer potentials of heat and mass for the sphere assume the following form:

$$T(x, \mathrm{Fo}) = 1 - (1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{\mathrm{Ki}_m}{\mathrm{Bi}_q} - \frac{1}{2}\,\varepsilon\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Ki}_m \times$$

$$\times\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right),$$

$$\Theta(x, \mathrm{Fo}) = \mathrm{Ki}_m\left[3\,\mathrm{Lu}\,\mathrm{Fo} - (1 + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu})\,\frac{3 - 5\,\frac{x^2}{R^2}}{10}\right].$$

CHAPTER 5

Two-dimensional Fields of Heat and Mass Transfer Potentials

§ 1. Finite Plate. Boundary Conditions of the Second Kind

Consider the solutions of a system of differential equations of heat and mass transfer applicable to a two-dimensional plate.

We have:

$$\frac{\partial t(x, y, \tau)}{\partial \tau} = a_q \left(\frac{\partial^2 t(x, y, \tau)}{\partial x^2} + \frac{\partial^2 t(x, y, \tau)}{\partial y^2} \right) + \\ + \frac{\varepsilon \varrho}{c_q} \frac{\partial u(x, y, \tau)}{\partial \tau}, \qquad (1.1)$$

$$\frac{\partial u(x, y, \tau)}{\partial \tau} = a_m \left(\frac{\partial^2 u(x, y, \tau)}{\partial x^2} + \frac{\partial^2 u(x, y, \tau)}{\partial y^2} \right) + \\ + a_m \delta \left(\frac{\partial^2 t(x, y, \tau)}{\partial x^2} + \frac{\partial^2 t(x, y, \tau)}{\partial y^2} \right), \qquad (1.2)$$

where $u = c'_T \theta$ is the specific mass content.

The initial conditions are:

$$t(x, y, 0) = f(x, y), \quad u(x, y, 0) = \varphi(x, y). \qquad (1.3)$$

The boundary conditions are:

$$\frac{\partial t(0, y, \tau)}{\partial x} = \frac{\partial t(x, 0, \tau)}{\partial y} = 0, \qquad (1.4)$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, y, \tau)}{\partial y} = 0. \qquad (1.5)$$

$$\frac{\partial t(h, y, \tau)}{\partial x} = \psi_1(y, \tau), \quad \frac{\partial t(x, d, \tau)}{\partial y} = \chi_1(x, \tau), \tag{1.6}$$

$$\frac{\partial u(h, y, \tau)}{\partial x} = \psi_2(y, \tau), \quad \frac{\partial u(x, d, \tau)}{\partial y} = \chi_2(x, \tau). \tag{1.7}$$

For the solution of this problem we make use of a double Fourier transformation in the region Γ

$$(-h \leqslant x \leqslant h), \ (-d \leqslant y \leqslant d):$$

$$F\{\varphi(x, y)\} = \int_{(\Gamma)}\int \exp\left(im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\, d\xi\, d\eta = \bar{\varphi}(m, n). \tag{1.8}$$

This transformation reduces the real function $\varphi(x, y)$, integrable in the region Γ, to the expression $\bar{\varphi}(m, n)$,which is a function of two integral arguments m and n.

The transition from this expression to the inversion is carried out by the formula

$$\varphi(x, y) \doteq \frac{1}{4hd}\sum_{m, n}^{-\infty, \infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\bar{\varphi}(m, n)$$

$$= F^{-1}\{\bar{\varphi}(m, n)\}. \tag{1.9}$$

If the function $\varphi_1(x, y)$ is integrable within the rectangle $(-2h \leqslant x \leqslant 2h)$, $(-2d \leqslant y \leqslant 2d)$ and the function $\varphi_2(x, y)$ is integrable in the region Γ, then the convolution of these functions will be equal to

$$\varphi_1 * \varphi_2(x, y) = \int_{(\Gamma)}\int \varphi_1(x - \xi, y - \eta)\, \varphi_2(\xi, \eta)\, d\xi, d\eta. \tag{1.10}$$

If the function $\varphi_1(x, y)$ is periodic with respect to the first argument with a period $2h$ and with respect to the second argument with a period $2d$, then we have the following law of convolution

$$F\{\varphi_1 * \varphi_2\} = F\{\varphi_1\}\, F\{\varphi_2\}. \tag{1.11}$$

It is appropriate to continue the functions of t and u in the intervals $(-d \leqslant y \leqslant d)$ and $(-h \leqslant x \leqslant +h)$ in even form, and their partial derivatives t'_x, t'_y, u'_x and u'_y in odd form,

which can be done by means of the conditions (1.4), (1.5). Then, assuming that F can commute with $\partial/\partial\tau$ after the application of the Fourier transformation, we obtain the system of two ordinary differential equations:

$$\frac{d\bar{i}(m,n,\tau)}{d\tau} = -a_q\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\bar{i}(m,n,\tau) +$$

$$+2a_q\left\{(-1)^m\int_{-d}^{d}\exp\left(-in\frac{\pi}{d}\eta\right)\psi_1(\eta,\tau)\,d\eta +\right.$$

$$\left.+(-1)^n\int_{-h}^{h}\exp\left(-im\frac{\pi}{d}\xi\right)\chi_1(\xi,\tau)\,d\xi\right\} +$$

$$+\frac{\varepsilon_0}{c_q}\frac{d\bar{u}(m,n,\tau)}{d\tau}, \qquad (1.12)$$

$$\frac{d\bar{u}(m,n,\tau)}{d\tau} = -a_m\pi^2\left(\frac{m^2}{h_2}+\frac{n^2}{d^2}\right)\bar{u}(m,n,\tau) - a_m\delta\pi^2\times$$

$$\times\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\bar{i}(m,n,\tau) + 2a_m\left\{(-1)^m\int_{-d}^{d}\times\right.$$

$$\times\exp\left(-in\frac{\pi}{d}\eta\right)\psi_2(\eta,\tau)\,d\eta + (-1)^n\int_{-h}^{h}\times$$

$$\left.\times\exp\left(-im\frac{\pi}{h}\xi\right)\chi_2(\xi,\tau)\,d\xi\right\} + 2\frac{\varepsilon_0}{c_q}\times$$

$$\times\left\{(-1)^m\int_{-d}^{d}\exp\left(-in\frac{\pi}{d}\eta\right)\psi_1(\eta,\tau)\,d\eta +\right.$$

$$\left.+(-1)^n\int_{-h}^{h}\exp\left(-im\frac{\pi}{h}\xi\right)\chi_1(\xi,\tau)\,d\xi\right\} \qquad (1.13)$$

with the initial conditions

$$\bar{i}(m,n,0)=\bar{f}(m,n)\,,\quad \bar{u}(m,n,0)=\bar{\varphi}(m,n)\,. \qquad (1.14)$$

The equations (1.12), (1.13) will be solved by the method of Laplace integral transformation. As a result we obtain the

following solution for $\bar{t}(m, n, \xi)$ and $\overline{u}(m, n, \tau)$:

$$\overline{t}(m,n,\tau)=\sum_{j=1}^{2}(-1)^{j+1}\left\{A_j\,\overline{\varphi}(m,n)\exp\left[-v_j^2\,a_q\,\mathrm{Lu}\,\pi^2\times\right.\right.$$

$$\times\left(\frac{m^2}{h^2}+\frac{h^2}{d^2}\right)\tau\Big]+B_j\,\overline{f}(m,n)\exp\Big[-a_q\,\mathrm{Lu}\,v_j^2\,\pi^2\times$$

$$\times\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\Big]+C_j\exp\left[-a_q\,\mathrm{Lu}\,v_j^2\,\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\right]\overset{\tau}{*}\times$$

$$\times\left[(-1)^m\int_{-d}^{d}\exp\left(in\frac{\pi}{d}\eta\right)\psi_1(\eta,\tau)\,d\eta+\right.$$

$$\times(-1)^n\int_{-h}^{h}\exp\left(-im\frac{\pi}{h}\right)\chi_1(\xi,\tau)\,d\xi\Big]+D_j\times$$

$$\times\exp\left[-a_q\,\mathrm{Lu}\,v_j^2\,\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\right]\overset{\tau}{*}\left[(-1)^m\int_{-d}^{d}\times\right.$$

$$\times\exp\left(-in\frac{\pi}{d}\eta\right)\psi_2(\eta,\tau)\,d\eta+(-1)^n\int_{-h}^{h}\times$$

$$\left.\left.\times\exp\left(-im\frac{\pi}{h}\xi\right)\chi_2(\xi,\tau)\,d\xi\Big]\right\},\qquad(1.15)$$

where the symbol $\overset{\tau}{*}$ denotes convolution with respect to τ from 0 to τ and v_j^2 is determined by the formulae (1.9), (1.10).

The constant coefficients are equal to:

$$A_j=\frac{\varepsilon\varrho}{c_q(v_1^2-v_2^2)},\quad B_j=\frac{v_j^2-1}{v_1^2-v_2^2},$$

$$C_j=\frac{2\,a_q\,[v_j^2(1+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu})-1]}{v_1^2-v_2^2},$$

$$D_j=\frac{2\,a_q\,\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,v_j^2}{\delta(v_1^2-v_2^2)}.\qquad(1.16)$$

The transform $\overline{u}(m, n, \tau)$ is expressed by the righthand side of the equality (1.15), where, instead of the coefficients A_j, B_j, C_j, D_j, we must substitute the corresponding coeffi-

cients A'_j, B'_j, C'_j, D'_j, determined by the following formulae:

$$A'_j = \frac{\nu_j^2 - \frac{1}{\mathrm{Lu}} - \varepsilon\,\mathrm{Ko}\,\mathrm{Pn}}{\nu_1^2 - \nu_2^2}, \quad B'_j = \frac{\delta}{\nu_1^2 - \nu_2^2},$$

$$C'_j = \frac{2 a_q \delta\,\mathrm{Lu}\,\nu_j^2}{\nu_1^2 - \nu_2^2}, \quad D'_j = \frac{2 a_q(\mathrm{Lu}\,\nu_j^2 - 1)}{\nu_1^2 - \nu_2^2}. \qquad (1.17)$$

Before we pass over to the inversion, it is necessary to note certain known properties of the theta-functions as functions of two variables:

$$\vartheta_3(\nu, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n}, \quad (z = e^{i\pi\nu}, q = e^{-i\pi\tau}). \qquad (1.18)$$

The function $\overline{K}(m, n, z)$ determined by the relation

$$\overline{K}(m, n, z) = \exp\left[-\pi^2\left(\frac{m^2}{h^2} + \frac{n^2}{d^2}\right)z\right], \qquad (1.19)$$

has, as its inversion,

$$K(x, y, z) = \frac{1}{4hd}\vartheta_3\left[\frac{x}{2h}, \frac{i\pi z}{h^2}\right]\vartheta_3\left[\frac{y}{2d}, \frac{i\pi z}{d^2}\right]. \qquad (1.20)$$

As a result of the known properties of the theta-functions, it follows, that the function $\overline{K}(x, y, z)$ is periodic with respect to x with a period $2h$, and with respect to y with a period $2d$. Taking this into account, and remembering the relations (1.19), (1.20), it is not difficult to obtain the inversion with respect to m and n.

After the final transformations we shall have

$$t(x, y, \tau) = \frac{1}{4hd}\sum_{j=1}^{2}(-1)^{j+1}\Big\{A_j \int\int_{(T)} \vartheta_3\left(\frac{x-\xi}{2h}, i\,\mathrm{Lu}, \nu_j^2\pi\,\mathrm{Fo}_1\right)\times$$

$$\times\,\vartheta_3\left(\frac{y-\eta}{2d}, i\,\mathrm{Lu}\,\pi\nu_j^2\,\mathrm{Fo}_2\right)\varphi(\xi, \eta)\,d\xi, d\eta +$$

$$+ B_j \int\int_{(T)} \vartheta_3\left(\frac{x-\xi}{2h}, i\pi\nu_j^2\,\mathrm{Lu}\,\mathrm{Fo}_1\right)\vartheta_3\left(\frac{y-\eta}{2d}, i\pi\nu_j^2\,\mathrm{Lu}\,\mathrm{Fo}_2\right)\times$$

$$\times\, f(\xi, \eta)\,d\xi\,d\eta + C_j \int_0^{\tau} d\,0\left(\vartheta_3\left[\frac{h-x}{2h}, i\pi\nu_j^2\,\mathrm{Lu}\,\mathrm{Fo}_1\left(1 - \frac{0}{\tau}\right)\right]\times\right.$$

$$\times \int_{-d}^{d} \vartheta_3 \left[\frac{y-\eta}{2d}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo}_2 \left(1-\frac{\theta}{\tau}\right)\right] \psi_1(\eta, \theta)\, d\eta +$$

$$+ \vartheta_3 \left[\frac{d-y}{2d}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo}_2 \left(1-\frac{\theta}{\tau}\right)\right] \int_{-h}^{h} \vartheta_3 \times$$

$$\times \left[\frac{x-\xi}{2\lambda}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo}_1 \left(1-\frac{\vartheta}{\theta}\right)\right] \chi_1(\xi, \theta)\, d\xi\Big) +$$

$$+ D_j \int_0^{\tau} d\theta \Big(\vartheta_3 \left[\frac{h-x}{2h}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo} \left(1-\frac{\theta}{\tau}\right)\right] \int_{-d}^{d} \vartheta_3 \times$$

$$\times \left[\frac{y-\eta}{2d}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo}_2 \left(1-\frac{\vartheta}{\tau}\right)\right] \psi_2(\eta, \vartheta)\, d\eta + \vartheta_3 \times$$

$$\times \left[\frac{d-y}{2d}, i\pi\nu^2 \operatorname{Lu} \operatorname{Fo}_2 \left(1-\frac{\tau}{\theta}\right)\right] \int_{-h}^{h} \vartheta_3 \left[\frac{x-\xi}{2h}, i\pi\nu_j^2 \operatorname{Lu} \operatorname{Fo}_1 \times\right.$$

$$\left.\times \left(1-\frac{\tau}{\theta}\right)\right] \chi_2(\xi, \tau)\, d\xi\Big)\Big\}. \tag{1.21}$$

The inversion $u(x, y, \tau)$ is obtained in the same way. As a result we shall obtain for $u(x, y, \tau)$ an expression analogous to the relation (1.21), but only the coefficients A_j, B_j, C_j and D_j need be replaced by A_j', B_j', C_j' and D_j'.

§ 2. Two-dimensional Plate. Generalized Boundary Conditions

By means of the simultaneous application of two-dimensional finite complex Fourier transformations and the Laplace transformation we can solve also the following series of problems, concerned with the non-isothermal transfer of mass in a plate of finite dimensions ($h \times d$) under generalized boundary conditions.

Problem 1. It is required to solve the system of differential equations (1.1) and (1.2) (see § 1) under the initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions

$$\frac{\partial l(0, y, \tau)}{\partial x} = \frac{\partial l(x, 0, \tau)}{\partial y} = 0\,, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, 0, \tau)}{\partial y} = 0\,, \tag{5}$$

$$-\lambda_q \frac{\partial l(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0\,, \tag{6}$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial l(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0\,, \tag{7}$$

$$-\lambda_q \frac{\partial l(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0\,, \tag{8}$$

$$a_m \frac{\partial u(x, d, \tau)}{\partial y} + a_m \delta \frac{\partial l(x, d, \tau)}{\partial y} + \chi_2(x, \tau) = 0\,. \tag{9}$$

To solve problem we shall make use of the two-dimensional finite complex Fourier transformation in the region

$$\Gamma(-h \leqslant x \leqslant +h, -d \leqslant y \leqslant +d)$$

$$\bar{\varphi}(m, n) = \int\int \exp\left(-im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\, d\xi\, d\eta \tag{10}$$

and of the corresponding inversion formula

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m,n)}^{-\infty,+\infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\bar{\varphi}(m, n)\,. \tag{11}$$

If

$$\bar{\varphi}_1(m, n) \approx \varphi_1(x, y)\,,$$
$$\bar{\varphi}_2(m, n) \approx \varphi_2(x, y)\,, \tag{12}$$

then, under the conditions

$$\varphi_1(x + 2h, y) = \varphi_1(x, y)\,,$$
$$\varphi_1(x, y + 2d) = \varphi_1(x, y)\,, \tag{13}$$

the following convolution theorem holds:

$$\bar{\varphi}_1(m, n)\,\bar{\varphi}_2(m, n) \approx \int_{(\Gamma)}\int \varphi_1(x - \xi, y - \eta)\,\varphi_2(\xi, \eta)\, d\xi\, d\eta\,. \tag{14}$$

Here it is assumed that $\varphi_2(x, y)$ integrates in the region Γ, and $\varphi_1(x, y)$ in the wider region $(-2h \leqslant x \leqslant 2h, -2d \leqslant y \leqslant 2d)$.

Substituting in (10) instead of φ the partial derivatives, and then carrying out integration by parts, we obtain formulae for the transforms of the partial derivatives:

$$\frac{\partial \varphi}{\partial x} \approx im \frac{\pi}{h} \bar{\varphi}(m, n) + (-1)^m \int_{-d}^{+d} \exp\left(-in \frac{\pi}{d} \eta\right) \times$$

$$\times [\varphi(h, \eta) + \varphi(-h, \eta)]\, d\eta, \tag{15}$$

$$\frac{\partial \varphi}{\partial y} \approx in \frac{\pi}{d} \bar{\varphi}(m, n) + (-1)^n \int_{-h}^{+h} \exp\left(-im \frac{\pi}{d} \eta\right) \times$$

$$\times [\varphi(\xi, d) + \varphi(\xi, -d)]\, d\xi, \tag{16}$$

$$\frac{\partial^2 \varphi}{\partial x^2} \approx -m^2 \frac{\pi^2}{h^2} \bar{\varphi}(m, n) + (-1)^m \int_{-d}^{+d} \exp\left(-in \frac{\pi}{d} \eta\right) \times$$

$$\times \left\{ \frac{\partial \varphi(h, \eta)}{\partial \xi} - \frac{\partial \varphi(-h, \eta)}{\partial \xi} + \right.$$

$$\left. + im \frac{\pi}{h} \left[\varphi(h, \eta) - \varphi(-h, \eta)\right] \right\} d\eta, \tag{17}$$

$$\frac{\partial^2 \varphi}{\partial y^2} \approx -n^2 \frac{\pi^2}{h^2} \bar{\varphi}(m, n) + (-1)^n \int_{-h}^{+h} \exp\left(-im \frac{\pi}{h} \xi\right) \times$$

$$\times \left\{ \frac{\partial \varphi(\xi, d)}{\partial \eta} - \frac{\partial \varphi(\xi, -d)}{\partial \eta} + \right.$$

$$\left. + in \frac{\pi}{d} [\varphi(\xi, d) - \varphi(\xi, -d)] \right\} d\xi. \tag{18}$$

Continuing now the functions t and u in even form in the intervals $(-h \leqslant x \leqslant 0)$ and $(-d \leqslant y \leqslant 0)$ and their partial derivatives $\partial t/\partial x$, $\partial t/\partial y$, $\partial u/\partial x$ and $\partial u/\partial y$ in odd form, which can be done by means of (4), (5), and passing in (1) – (9) to the transforms relative to x and y by means of the formulae (15) – (18) we obtain instead of the system of partial differential equations (1), (2), a system of two ordinary differential equations for the functions $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{\imath}(m,n,\tau)}{d\tau} &= -a_q\sigma\,\bar{\imath}(m,n,\tau)+\frac{2a_q}{\lambda_q}S_1(\tau)+ \\ &\quad +\frac{\varepsilon\varrho}{c}\,\frac{d\bar{u}(m,n,\tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m\sigma\,\bar{u}(m,n,\tau)-2S_2(\tau)- \\ &\quad -a_m\delta\sigma\,\bar{\imath}(m,n,\tau) \end{aligned}\right\}, \tag{19}$$

where

$$\sigma=\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right), \tag{20}$$

$$S_j(\tau)=(-1)^m\int_{-d}^{+d}\exp\left(-in\frac{\pi}{d}\eta\right)\psi_j(\eta,\tau)\,d\eta+$$

$$+(-1)^n\int_{-h}^{+h}\exp\left(-im\frac{\pi}{h}\xi\right)\chi_j(\xi,\tau)\,d\xi. \tag{21}$$

The initial transformed conditions (3) take the form

$$\bar{\imath}(m,n,0)=\bar{f}_1(m,n),\ \ \bar{u}(m,n,0)=\bar{f}_2(m,n). \tag{22}$$

Now, passing in (19) to the Laplace transforms relative to τ, we obtain a system of two algebraic equations of the first degree for the two unknown functions $\bar{\imath}(m,\ n,\ s)$ and $\bar{u}(m,\ n,\ s)$. Solving this system and passing over to the original relative to s, we obtain

$$\imath(m,n,\tau)=\sum_{l=1}^{2}\{\exp(-a_q\,\mathrm{Lu}\,\nu_l^2\,\sigma\tau)\,[A_l\bar{f}_1(m,n)+B_l\bar{f}_2(m,n)]+$$

$$+\exp(-a_q\,\mathrm{Lu}\,\nu_l^2\,\sigma\tau)\,[D_l\overset{\tau}{*}S_2(\tau)+C_l\overset{\tau}{*}S_1(\tau)]\}. \tag{23}$$

Here, the symbol $\overset{\tau}{*}$ denotes convolution with respect to τ from 0 to τ. The coefficients A_l, B_l, C_l, D_l are determined from the formulae:

$$A_l=(-1)^{l+1}\frac{\nu_l^2-1}{\nu_1^2-\nu_2^2},\ \ B_l=(-1)^{l+1}\frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}}{\nu_1^2-\nu_2^2},$$

$$C_l=\frac{2}{c_q\gamma_0}A_l,\ \ D_l=2\nu_l^2B_l. \tag{24}$$

The second function $\overline{u}(m, n, \tau)$ is obtained from (23), if, on the righthand side, we replace the coefficients A_i, B_i, C_i, D_i by the coefficients A'_i, B'_i, C'_i, D'_i, having the following form:

$$A'_i = (-1)^{i+1} \frac{\delta}{\nu_1^2 - \nu_2^2},$$

$$B'_i = \frac{(-1)^{i+1}}{\nu_1^2 - \nu_2^2} \left(\nu_i^2 - \varepsilon \,\mathrm{Ko}\,\mathrm{Pn} - \frac{1}{\mathrm{Lu}} \right),$$

$$C'_i = (-1)^{i+1} \frac{2\,\delta}{c_q \gamma_0 (\nu_1^2 - \nu_2^2)},$$

$$D'_i = (-1)^{i+1} \frac{\left(\nu_i^2 - \dfrac{1}{\mathrm{Lu}} \right)}{\nu_1^2 - \nu_2^2}. \tag{25}$$

In order to obtain the final solution of the problem, it remains to invert (23) with respect to m and n. This transition may be carried out with the help of the convolution theorem, making use of theta-functions. Remember that, by definition,

$$\vartheta_3(v, \tau) = \sum_{n=-\infty}^{+\infty} q^{n^2} z^{2n} (z = e^{i\pi v}, \;\; q = e^{i\pi\tau}). \tag{26}$$

Consequently, according to (11), (20), and (26),

$$e^{-\sigma\zeta} \approx \frac{1}{4hd} \vartheta_3 \left(\frac{x}{2h}, \frac{i\pi\zeta}{h^2} \right) \vartheta_3 \left(\frac{y}{2d}, \frac{i\pi\zeta}{d^2} \right). \tag{27}$$

Here, on the right hand side, we have a function periodic with respect to x with a period $2h$ and with respect to y with a period $2d$. Keeping this in mind, and remembering (25), we pass over to the originals of the first two components on the right hand side of (23) by the convolution theorem (14), and for the following components immediately by formula (11). As a result we obtain

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + \\ + C_1 K_{11} + C_2 K_{21} + D_1 K_{12} + D_2 K_{22}, \tag{28}$$

where

$$I_{mn} = \frac{1}{4\,h\,d} \int\limits_{(T)}\int \vartheta_3\left(\frac{x-\xi}{2\,h}, \frac{i\,\pi\,\mathrm{Lu}\,a_q\,r_i^2\,\tau}{h^2}\right) \times$$

$$\times\, \vartheta_3\left(\frac{y-h}{2\,d}, \frac{i\,\pi\,\mathrm{Lu}\,a_q\,r_m^2\,\tau}{d^2}\right) f_n(\xi,\eta)\,d\xi\,d\eta\,, \qquad (29)$$

$$K_{mn} = \frac{1}{4\,hd} \int\limits_0^\tau d\,\theta \Big\{ \vartheta_3\left(\frac{h-x}{2\,h}, \frac{i\,\pi\,\mathrm{Lu}\,a_q\,r_m^2(\tau-\theta)}{h^2}\right) \times$$

$$\times \int\limits_{-d}^{+d} \vartheta_3\left(\frac{y-h}{2\,d}, \frac{i\,\pi\,a_q\,\mathrm{Lu}\,r_m^2(\tau-\theta)}{d^2}\right) \psi_n(\eta,\theta)\,d\,\eta +$$

$$+ \vartheta_3\left(\frac{d-y}{2\,d}, \frac{i\,\pi\,\mathrm{Lu}\,a_q\,r_m^2(\tau-\theta)}{d^2}\right) \times$$

$$\times \int\limits_{-c}^{+c} \vartheta_3\left(\frac{x-\xi}{2\,h}, \frac{i\,\pi\,\mathrm{Lu}\,a_q\,r_m^2(\tau-\theta)}{h^2}\right) \chi_n(\xi,\theta)\,d\,\xi \Big\}\,. \qquad (30)$$

If we introduce the symbols

$$A_1 = A_{11},\ A_2 = A_{21},\ B_1 = A_{12},\ B_2 = A_{22},\ C_1 = B_{11}\,,$$

$$C_2 = B_{21},\ D_1 = B_{12},\ D_2 = B_{22}\,,$$

then formula (28) can be rewritten in the form

$$l(x,y,\tau) = \sum_{i,j} (A_{ij}\,I_{ij} + B_{ij}\,K_{ij})\,, \qquad (31)$$

$$i,j = 1,2\,.$$

By virtue of the observation already made with regard to the function $\bar{u}(m,\ n,\ \tau)$, its inversion will have the form

$$u(x,y,\tau) = \sum_{(i,j)} (A'_{ij}\,I_{ij} + B'_{ij}\,K_{ij})\,(i,j = 1,2)\,. \qquad (32)$$

Problem 2. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$l(x,y,0) = f_1(x,y),\ u(x,y,0) = f_2(x,y) \qquad (3)$$

and the boundary conditions

$$\frac{\partial l(0,y,\tau)}{\partial x} = \frac{\partial l(x,0,\tau)}{\partial y} = 0\,, \qquad (4)$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, 0, \tau)}{\partial y} = 0\,, \tag{5}$$

$$t(h, y, \tau) = \varphi_1(y, \tau)\,, \tag{6}$$

$$u(h, y, \tau) = \varphi_2(y, \tau)\,, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0\,, \tag{8}$$

$$a_m \left(\frac{\partial u(x, d, \tau)}{\partial y} + \delta \frac{\partial t(x, d, \tau)}{\partial y}\right) + \chi_2(x, \tau) = 0\,. \tag{9}$$

We make use of the two-dimensional finite complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant h,\ -d \leqslant y \leqslant d)$:

$$\bar{\varphi}(m, n) = \int\limits_{(\Gamma)}\int \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi - \right.$$

$$\left. - in\,\frac{\pi}{d}\,\eta\right] \varphi(\xi, \eta)\, d\xi\, d\eta\,, \tag{10}$$

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m,n)}^{-\infty,+\infty} \exp\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}x + \right.$$

$$\left. + in\,\frac{\pi}{d}\,y\right] \bar{\varphi}(m, n)\,. \tag{11}$$

By applying the transformation (10) to the equations (1.1) and (1.2) we obtain a system of two ordinary differential equations for the functions $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{t}(m,n,\tau)}{d\tau} &= -a_q \sigma\, \bar{t}(m, n, \tau) + 2a_q S_1(\tau) + \frac{2a_q}{\lambda_q} I_1(\tau) + \\ &\quad + \frac{\varepsilon\varrho}{c}\,\frac{d\bar{u}(m, n, \tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m \sigma\, \bar{u}(m, n, \tau) + \\ + 2a_m S_2(\tau) - 2J_2(\tau) - a_m \delta\sigma\, \bar{u}(m, n, \tau) &+ 2a_m \delta\, S_1(\tau) \end{aligned}\right\}, \tag{12}$$

where

$$\left[\left(m - \frac{1}{2}\right)^2 \right.$$

$$\left.\begin{aligned} S_l(\tau) &= (-1)^m i \int_{-d}^{+d} \exp\left(-in\frac{\pi}{d}\eta\right) \times \\ &\times \left[i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\right] \psi_j(\eta,\tau)\, d\eta \\ I_j(\tau) &= (-1)^n \int_{-h}^{+h} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi\right] \chi_j(\xi,\tau)\, d\xi \end{aligned}\right\}. \quad (14)$$

Solving this system and passing over to the inversions with respect to (s) we obtain

$$\bar{l}(m,n,\tau) = \sum_{i=1}^{2} \{\exp(-a_q \operatorname{Lu} \nu_i^2 \sigma\tau)[A_i \bar{f}_1(m,n) +$$

$$+ B_i \overline{f_2}(m,n) + C_i^* I_1(\tau) + D_i^* I_2(\tau) + a_m D_i^* S_2(\tau) +$$

$$+ E_i^* S_1(\tau)]\}. \quad (15)$$

The coefficients A_l, B_l, C_l, D_l are determined from the formulae (25) of the previous problem, and the coefficients E_l from the formula

$$E_l = (-1)^{l+1} \frac{2a_q}{\nu_1^2 - \nu_2^2}[\nu_l^2(1 + \varepsilon \operatorname{Lu} \operatorname{Ko} \operatorname{Pn}) - 1]. \quad (16)$$

The second function $\bar{u}(m, n, \tau)$ is obtained from (15), if, on the right hand side, we replace the coefficients A_l, B_l, C_l, D_l, E_l by the coefficients A'_l, B'_l, C'_l, D'_l and E'_l, the last having the form

$$E'_l = 2a_m \delta(-1)^{l+1} \frac{\nu_1^2}{\nu_1^2 - \nu_2^2}. \quad (17)$$

Making use of the convolution theorem and the inversion formula (11), we finally obtain

$$l(x,y,\tau) + A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + E_1 K_{11} +$$

$$+ E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22} + C_1 M_{11} + C_2 M_{21} +$$

$$+ D_1 M_{12} + D_2 M_{22}, \quad (18)$$

where

$$I_{kl} = \frac{1}{4hd} \int\limits_{(\Gamma)} \int \vartheta_2 \left(\frac{x-\xi}{2h}, \ \frac{a_k i \pi \tau}{h^2} \right) \vartheta_3 \left(\frac{y-\eta}{2d}, \ \frac{a_k i \pi \tau}{d^2} \right) \times$$

$$\times f_l(\xi, \eta) \, d_\xi d\eta \, , \tag{19}$$

$$K_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \frac{\partial}{\partial x} \vartheta_2 \left[\frac{h-x}{2h}, \ \frac{a_k i \pi(\tau-\theta)}{h^2} \right] \times \right.$$

$$\left. \times \int\limits_{-d}^{+d} \vartheta_3 \left[\frac{y-\eta}{2d}, \ \frac{a_k i \pi(\tau-\theta)}{d^2} \right] \psi_l(\eta, d) \, d_\eta \right\} , \tag{20}$$

$$M_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \vartheta_3 \left[\frac{d-y}{2d}, \ \frac{a_k i \pi(\tau-\theta)}{d^2} \right] \times \right.$$

$$\left. \times \int\limits_{-h}^{+h} \vartheta_2 \left[\frac{x-\xi}{2h}, \ \frac{a_k i \pi(\tau-\theta)}{h^2} \right] \chi_l(\xi \, d) \, d_\xi \right\} , \tag{21}$$

$$a_k = v_k^2 \operatorname{Lu} a_q, \ k, l = 1, 2 \, .$$

By virtue of the observation already made relating to the function $\bar{u}(m, n, \tau)$, its inversion is expressed by the right hand side of the formula (18) in which $A_i \ldots$ must be replaced by $A_i' \ldots$.

Problem 3. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y \, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x} = \frac{\partial t(x, 0, \tau)}{\partial y} = 0 \, , \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, 0, \tau)}{\partial y} = 0 \, , \tag{5}$$

$$t(h, y, \tau) = \psi_1(y, \tau) \, , \tag{6}$$

$$u(h, y, \tau) = \psi_2(y, \tau) \, , \tag{7}$$

$$t(x, d, \tau) = \chi_1(x, \tau) \, , \tag{8}$$

For the solution of the problem we make use of the two-dimensional finite complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant h,\ -d \leqslant y \leqslant d)$:

$$\bar{\varphi}(m,n) = \int\int_{(\Gamma)} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi - \right.$$
$$\left. - i\left(n-\frac{1}{2}\right)\frac{\pi}{d}\eta\right]\varphi(\xi,\eta)\,d\xi\,d\eta \qquad (10)$$

and of the corresponding inversion formula

$$\varphi(x,y) = \frac{1}{4hd}\sum_{(m,n)}^{-\infty,+\infty} \exp\left[i\left(m-\frac{1}{2}\right)\frac{\pi}{h}x + \right.$$
$$\left. + i\left(n-\frac{1}{2}\right)\frac{\pi}{d}y\right]\bar{\varphi}(m,n). \qquad (11)$$

If

$$\bar{\varphi}_1(m,n) \approx \varphi_1(x,y), \qquad (12)$$
$$\bar{\varphi}_2(m,n) \approx \varphi_2(x,y),$$

then, under the condition

$$\varphi_1(x \pm 2h, y) = -\varphi_1(x,y), \qquad (13)$$
$$\varphi_1(x, y \pm 2d) = -\varphi_1(x,y),$$

the convolution theorem holds:

$$\bar{\varphi}_1(m,n)\,\bar{\varphi}_2(m,n) \approx \int\int_{(\Gamma)} \varphi_1(x-\xi, y-\eta)\,\varphi_2(\xi,\eta)\,d\xi\,d\eta. \qquad (14)$$

Substituting in (10) instead of φ the partial derivatives and then carrying out integration by parts, we obtain a system of two ordinary differential equations for the functions $\bar{i}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{i}(m,n,\tau)}{d\tau} &= -a_q\sigma\,\bar{i}(m,n,\tau) \pm a_q S_1(\tau) \pm \frac{\varepsilon_0}{c}\frac{d\bar{u}(m,n,\tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m\sigma\,\bar{u}(m,n\tau) \pm a_m S_2(\tau) - \\ &\quad - a_m\delta\bar{\sigma}\,\bar{i}(m,n,\tau) \pm a_m S_1(\tau) \end{aligned}\right\}, \qquad (15)$$

where

$$\sigma = \frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2}\,, \tag{16}$$

$$S_j(\tau) = 2(-1)^m i \int_{-d}^{+d} \exp\left[-i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right]\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\right] \times$$

$$\times\, \psi_1(\eta, \tau)\, d\eta + 2(-1)^n i \int_{-h}^{+h} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi\right] \times$$

$$\times \left[i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\right] \chi_1(\xi, \tau)\, d\xi\,. \tag{17}$$

The final solution of equation (15) has the form

$$t(x, y\,\tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + 2E_1 K_{11} +$$

$$+ 2E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22}\,, \tag{18}$$

where

$$I_{kl} = \frac{1}{4hd} \iint_{(\Gamma)} \vartheta_2\left(\frac{x - \xi}{2h}, \frac{i\pi a_k \tau}{h^2}\right) \vartheta_2\left(\frac{y - \eta}{2d}, \frac{i\pi a_k \tau}{d^2}\right) \times$$

$$\times\, f_l(\xi, \eta)\, d\xi, d\eta\,, \tag{19}$$

$$K_{kl} = \frac{1}{2hd} \int_0^{\tau} d\theta \left\{\frac{\partial}{\partial x}\vartheta_2\left[\frac{h - x}{2h}, \frac{i\pi a_k(\tau - \theta)}{h^2}\right] \times \right.$$

$$\left. \times \int_{-d}^{+d} \vartheta_2\left[\frac{y - \eta}{2d}, \frac{i\pi a_k(\tau - \theta)}{d^2}\right] \psi l(\eta, \theta)\, d\eta\right\} +$$

$$+ \frac{1}{2hd} \int_0^{\tau} d\theta \left\{\frac{\partial}{\partial x}\vartheta_2\left[\frac{d - y}{2d}, \frac{i\pi a_k(\tau - \theta)}{d^2}\right] \times \right.$$

$$\left. \times \int_{-h}^{+h} \vartheta_2\left[\frac{x - \xi}{2h}, \frac{i\pi a_k(\tau - \theta)}{h^2}\right] \chi_l(\xi, a)\, d\xi\right\}, \tag{20}$$

$$a_k = a_q \,\mathrm{Lu}\, \nu_k^2\,.$$

The inversion of the function $\bar{u}(m, n, \tau)$ is expressed by the right hand side of Formula (18), in which $A_l \ldots$ must be replaced by $A_l' \ldots$.

Problem 4. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions:

$$\frac{\partial t(0, y\tau)}{\partial x} = t(x, 0, \tau) = 0\,, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = u(x, 0, \tau) = 0\,, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0\,, \tag{6}$$

$$a_m\left[\frac{\partial u(h, y, \tau)}{\partial x} + \delta \frac{\partial t(h, y, \tau)}{\partial x}\right] + \psi_2(y, \tau) = 0\,, \tag{7}$$

$$t(x, d, \tau) = \chi_1(x, \tau)\,, \tag{8}$$

$$u(x, d, \tau) = \chi_2(x, \tau)\,. \tag{9}$$

To solve the problem we make use of the two-dimensional finite complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant h,\ -d \leqslant y \leqslant d)$:

$$\overline{\varphi}(m, n) = \int\limits_{(\Gamma)}\int \exp\left(-im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\,d\xi\,d\eta \tag{10}$$

and the corresponding inversion formula:

$$\varphi(x, y) = \frac{1}{4hd}\sum_{(m,n)}^{+\infty,-\infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\overline{\varphi}(m, n)\,. \tag{11}$$

Extending the functions τ and u in the interval $(-h \leqslant x \leqslant 0)$ in even form, and in the interval $(-d \leqslant y \leqslant 0)$ in the odd form, which can be done by virtue of (4), (5), we obtain

$$t(x, y, \tau) = t(-x, y, \tau),\ u(x, y, \tau) = u(-x, y, \tau)\,, \tag{12}$$

$$t(x, y, \tau) = -t(x, -y, \tau),\ u(x, y, \tau) = -u(x, -y, \tau)\,, \tag{13}$$

$$\frac{\partial t(x, y, \tau)}{\partial x} = -\frac{\partial t(-x, y, \tau)}{\partial x}\,,$$

$$\frac{\partial u(x, y, \tau)}{\partial x} = -\frac{\partial u(-x, y, \tau)}{\partial x}\,, \tag{14}$$

$$\frac{\partial t(x, y, \tau)}{\partial y} = \frac{\partial t(x, -y, \tau)}{\partial y},$$

$$\frac{\partial u(x, y, \tau)}{\partial y} = \frac{\partial u(x, -y, \tau)}{\partial y}, \tag{15}$$

$$f_1(x, y) = f_1(-x, y) = -f_1(x, y),$$

$$f_2(x, y) = f_2(-x, y) = -f_2(x, y), \tag{16}$$

$$\psi_1(y, \tau) = -\psi_1(-y, \tau), \quad \psi_2(y, \tau) = -\varphi_2(-y, \tau), \tag{17}$$

$$\chi_1(x, \tau) = \chi_1(-x, \tau), \qquad \chi_2(x, \tau) = \chi_2(-x, \tau). \tag{18}$$

Passing over now by means of (10) to the transforms relative to x and y in (1) – (9), we obtain, instead of (1.1), (1.2) a system of two ordinary differential equations for the functions $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$.

Finally, we obtain

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + C_1 K_{11} + \\ + C_2 K_{21} + D_1 K_{11} + D_2 K_{22} + E_1 M_{11} + E_2 M_{21} + \\ + a_m D_1 M_{12} + a_m D_2 M_{22}, \tag{19}$$

$$I_{kl} = \frac{1}{4hd} \int\limits_{(\Gamma)} \int \vartheta_3 \left(\frac{x - \xi}{2h}, \frac{i \pi a_k \tau}{h^2} \right) \times \\ \times \vartheta_3 \left(\frac{y - \eta}{2d}, \frac{i \pi a_k \tau}{d^2} \right) f_l(\xi, \eta) \, d\xi \, d\eta,$$

$$K_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \vartheta_3 \left[\frac{h - x}{2h}, \frac{i \pi a_k (\tau - \theta)}{h^2} \right] \times \right. \\ \left. \times \int\limits_{-d}^{+d} \vartheta_3 \left[\frac{y - \eta}{2d}, \frac{i \pi a_s (\tau - \theta)}{d^2} \right] \psi_l(\eta, \alpha) \, d\eta \right\},$$

$$M_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \frac{\partial}{\partial y} \vartheta_3 \left[\frac{d - y}{2d}, \frac{i \pi a_k (\tau - \theta)}{d^2} \right] \times \right. \\ \left. \times \int\limits_{-h}^{+h} \vartheta_3 \left[\frac{x - \xi}{2h}, \frac{i \pi a_k (\tau - \theta)}{h^2} \right] \chi_l(\xi, 0) \, d\xi \right\},$$

$$a_k = d_q \, \mathrm{Lu} \, \nu_k^2.$$

Problem 5. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions

$$t(x, y, 0) = f_1(x, y). \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions:

$$\frac{\partial t(0, y, \tau)}{\partial x} = t(x, 0, \tau) = 0\,, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = u(x, 0, \tau) = 0\,, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{dx} + \psi_1(y, \tau) = 0\,, \tag{6}$$

$$a_m \left[\frac{\partial u(h, y, \tau)}{\partial x} + \delta \frac{\partial t(h, y, \tau)}{\partial x}\right] + \psi_2(y, \tau) = 0\,, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0\,, \tag{8}$$

$$a_m \left[\frac{\partial u(x, d, \tau)}{\partial y} + \delta \frac{\partial t(x, d, \tau)}{\partial y}\right] + \chi_2(x, \tau) = 0\,. \tag{9}$$

For the solution of the problem we make use of the two-dimensional finite complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant h,\ -d \leqslant y \leqslant d)$:

$$\bar{\varphi}(m, n) = \int\limits_{(\Gamma)}\int \exp\left[-im\frac{\pi}{h}\xi - i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right] \times$$

$$\times\, \varphi(\xi, \eta)\, d\xi\, d\eta \tag{10}$$

and the corresponding inversion formula

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m,n)}^{-\infty,+\infty} \exp\left[im\frac{\pi}{h}x + i \times\right.$$

$$\left. \times \left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right] \bar{\varphi}(m, n)\,. \tag{11}$$

Passing over to the Laplace transform relative to τ, we obtain a system of two algebraic equations of the first degree for the two unknown functions $\bar{t}(m, n, s)$ and $\bar{u}(m, n, s)$. Solving this system and passing to the inversion, relative to s, we obtain for $t(m, n, \tau)$ and $\bar{u}(m, n, \tau)$ the same expressions as in

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0\,, \tag{8}$$

$$a_m \frac{\partial u(x, d, \tau)}{\partial y} + a_m \delta \frac{\partial t(x, d, \tau)}{\partial y} + \chi_2(x, \tau) = 0\,. \tag{9}$$

For the solution of the problem we make use of inversion formulae (10) and (11) of Problem 3. Continuing t and u in the interval $(-h \leqslant x \leqslant 0)$ in even form, and in the interval $(-d \leqslant y \leqslant 0)$ in the odd form, and passing in (1) – (9) to the transforms relative to the space coordinates. we obtain the system of two ordinary differential equations (12) of Problem 2, where now

$$\sigma = \pi^2 \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2} \right], \tag{10}$$

$$S_j(\tau) = (-1)^m i \int_{-d}^{+d} \exp\left[-i \left(n - \frac{1}{2}\right) \frac{\pi}{d} \eta \right] \times$$

$$\times \left[i \left(m - \frac{1}{2}\right) \frac{\pi}{h} \right] \psi_j(\eta, \tau)\, d\eta\,, \tag{11}$$

$$I_j(\tau) = (-1)^n i \int_{-h}^{+h} \exp\left[-i \left(m - \frac{1}{2}\right) \frac{\pi}{h} \xi \right] \chi_j(\xi, \tau)\, d\xi\,. \tag{12}$$

The solution of this system is given by formula (18) of Problem 2:

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + E_1 K_{11} +$$

$$+ E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22} + C_1 M_{11} +$$

$$+ C_2 M_{21} + D_1 M_{12} + D_2 M_{22}\,, \tag{13}$$

where I_{kl}, K_{kl}, M_{kl} are obtained from formulae (19–21) of Problem 2.

Problem 7. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions:

$$\frac{\partial t(0, y, \tau)}{\partial x} = t(x, 0, \tau) = 0\,. \tag{4}$$

The solution of this system is carried out in the ordinary way.

Passing to the inversion on the basis of the inversion formula and the theorem of convolution, we obtain

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_2 I_{21} B_2 I_{22} + 2 E_1 K_{11} + \\ + 2 E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22}, \qquad (14)$$

where I_{kl}, K_{kl} are determined by formulae (19), (20), of Problem 3.

The coefficient A_l is given in Problem 3. The function $u(x, y, \tau)$ is expressed by the right hand side of formula (14) in which A_l, B_l ... must be replaced by A'_l, B'_l as in Problem 3.

Problem 8. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \qquad (3)$$

and the boundary conditions:

$$t(0, y, \tau) = t(x, 0, \tau) = 0, \qquad (4)$$

$$u(0, y, \tau) = u(x, 0, \tau) = 0, \qquad (5)$$

$$t(x, d, \tau) = \chi_1(x, \tau), \qquad (6)$$

$$u(x, d, \tau) = \chi_2(x, \tau), \qquad (7)$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0, \qquad (8)$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial t(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0. \qquad (9)$$

For the solution of the problem we continue the functions t and u in the intervals $(-h \leqslant x \leqslant 0)$ and $(-d \leqslant y \leqslant 0)$ in the odd form and make use of the inversion formulae of Problem 2. Passing in (1) – (9) to the transforms with respect to the space coordinates, we obtain the system of two ordinary differential equations of Problem 2 where now

$$\sigma = \pi^2 \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{n^2}{d^2} \right], \qquad (10)$$

$$S_j(\tau) = (-1)^n \int_{-h}^{+h} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi\right] \times$$

$$\times \left(in\,\frac{\pi}{d}\right) \chi_1(\xi, \tau)\, d\,\xi\,, \tag{11}$$

$$I_j(\tau) = (-1)^m\, i \int_{-d}^{+d} \exp\left(-in\,\frac{\pi}{d}\,\eta\right) \psi_1(\eta, \tau)\, d\,\eta\,. \tag{12}$$

Solving this system and passing to the inversions, we obtain

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} +$$
$$+ B_2 I_{22} + E_1 K_{11} + E_2 K_{21} +$$
$$+ a_m D_1 K_{12} + a_m D_2 K_{22} + C_1 M_{11} + C_2 M_{21} +$$
$$+ D_1 M_{12} + D_2 M_{22}\,, \tag{13}$$

where

$$I_{kl} = \frac{1}{4\,hd} \int\int_{(T)} \vartheta_2\left(\frac{x-\xi}{2\,h}, \frac{a_k\, i\,\pi\tau}{h^2}\right) \times$$
$$\times\, \vartheta_3\left(\frac{y-\eta}{2\,d}, \frac{a_k i\,\pi\tau}{d^2}\right) f_l(\xi, \eta)\, d_\xi\, d_\eta\,, \tag{14}$$

$$K_{kl} = \frac{1}{4\,hd} \int_0^\tau d\,\theta \left\{ \frac{\partial}{\partial y}\, \vartheta_3\left[\frac{d-y}{2\,d}\;\frac{i\,\pi a_k(\tau-\theta)}{d^2}\right] \times \right.$$
$$\left. \times \int_{-h}^{+h} \vartheta_2\left[\frac{x-\xi}{2\,h}, \frac{i\,\pi\,a_k(\tau-\theta)}{h^2}\right] \chi_l(\xi, \theta)\, d\,\xi \right\}, \tag{15}$$

$$M_{kl} = \frac{1}{4\,hd} \int_0^\tau d\theta \left\{ \vartheta_2\left[\frac{h-x}{2\,h}, \frac{i\,\pi\,a_k(\tau-\theta)}{h^2}\right] \times \right.$$
$$\left. \times \int_{-d}^{+d} \vartheta_3\left[\frac{y-\eta}{2\,d}, \frac{i\,\pi a_k(\tau-\theta)}{d^2}\right] \psi_l(\eta, \theta)\, d_\eta \right\}, \tag{16}$$

$$a_k = v_k^2\, a_q\, \mathrm{Lu}\,.$$

The coefficients $A_i \ldots$ are indicated in Problem 2. The function $u(x, y, \tau)$ is expressed by the right hand side of

Formula (13) where $A_l, B_l \ldots$ must be replaced by $A'_l, B'_l \ldots$ as in Problem 2.

Problem 9. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y, 0) = f_1(x, y), u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions:

$$t(0, y, \tau) = t(x, 0, \tau) = 0\,, \tag{4}$$

$$u(0, y, \tau) = u(x, 0, \tau) = 0\,, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0\,, \tag{6}$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial t(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0\,, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0\,, \tag{8}$$

$$a_m \frac{\partial u(x, d, \tau)}{\partial y} + a_m \delta \frac{\partial t(x, d, \tau)}{\partial y} + \chi_2(x, \tau) = 0\,. \tag{9}$$

For the solution of the problem with boundary conditions of the first and second kinds we make use of the transformation and inversion formulae:

$$\bar{\varphi}(m, n) = \int\limits_{(T)}\int \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi - \right.$$

$$\left. - i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right]\varphi(\xi, \eta)\, d\xi, d\eta\,, \tag{10}$$

Passing in (1) – (9) to transforms with respect to the space coordinates, we obtain a system of two ordinary differential equations as in Problem 1, where now

$$\varphi(x, y) = \frac{1}{4hd} \sum_{m,n}^{-\infty,+\infty} \exp\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}x + \right.$$

$$\left. + i\left(n - \frac{1}{2}\right)\frac{\pi}{d}y\right]\bar{\varphi}(m, n)\,. \tag{11}$$

$$\sigma = \left[\frac{\left(m-\frac{1}{2}\right)^2}{h^2} + \frac{\left(n-\frac{1}{2}\right)^2}{d^2}\right], \tag{12}$$

$$s_j(\tau) = (-1)^m i \int_{-d}^{+d} \exp\left[-i\left(n-\frac{1}{2}\right)\frac{\pi}{d}\eta\right]\psi_1(\eta,\tau)\,d\eta + \\ + (-1)^n i \int_{-h}^{+h} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi\right]\chi_1(\xi,\tau)\,d\xi. \tag{13}$$

Solving this system and passing over to the originals, we obtain

$$t(x,y,\tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + C_1 K_{11} + \\ + C_2 K_{12} + D_1 K_{12} + D_2 K_{22}, \tag{14}$$

where

$$I_{kl} = \frac{1}{4hd}\int\int_{(\Gamma)} \vartheta_2\left(\frac{x-\xi}{2h}, \frac{i\pi a_k \tau}{h^2}\right)\times \\ \times \vartheta_2\left(\frac{y-\eta}{2d}, \frac{i\pi a_k \tau}{d^2}\right) f_l(\xi,\eta)\,d\xi\,d\eta, \tag{15}$$

$$K_{kl} = \frac{1}{4hd}\int_0^\tau d\theta \left\{ \vartheta_2\left[\frac{h-x}{2h}, \frac{i\pi a_k(\tau-\theta)}{h^2}\right]\times \right. \\ \times \int_{-d}^{+d} \vartheta_2\left[\frac{y-\eta}{2d}, \frac{i\pi a_k(\tau-\theta)}{d^2}\right]\psi_l(\eta,\theta)\,d\eta + \\ + \vartheta_2\left[\frac{d-y}{2d}, \frac{i\pi a_k(\tau-\theta)}{d^2}\right]\times \\ \left. \times \int_{-h}^{+h} \vartheta_2\left[\frac{x-\xi}{2h}, \frac{i\pi a_k(\tau-\theta)}{h^2}\right]\chi_l(\xi,\theta)\,d\xi \right\}. \tag{16}$$

The coefficients $A_l \ldots$ are indicated in Problem 1. The) function $u(x, y, \tau)$ is expressed by the right-hand side of Formula (14) in which A_i, B_i must be replaced by A_i', B_i' from Problem 1.

Problem 10. It is required to solve the system of differential equations (1.1) and (1.2) under the initial conditions:

$$t(x, y, 0) = f_1(x, y)\,, \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and the boundary conditions:

$$t(0, y, \tau) = t(x, 0, \tau) = 0\,, \tag{4}$$

$$u(0, y, \tau) = u(x, 0, \tau) = 0\,, \tag{5}$$

$$t\,(h, y, \tau) = \psi_1(y, \tau)\,, \tag{6}$$

$$u(h, y, \tau) = \psi_2(y, \tau)\,, \tag{7}$$

$$t\,(x, d, \tau) = \chi_1(x, \tau)\,, \tag{8}$$

$$u(x, d, \tau) = \chi_2(x, \tau)\,. \tag{9}$$

For the solution of the problem with boundary conditions of the first kind we make use of the inversion formulae of Problem 1. Passing in (1) – (9) to transforms with respect to the space coordinates, we obtain the system of two ordinary differential equations of Problem 3, where now

$$\sigma = \pi^2 \left(\frac{m^2}{h^2} + \frac{n^2}{d^2}\right), \tag{10}$$

$$S_j(\tau) = 2\,(-1)^m \int_{-d}^{+d} \exp\left(-in\frac{\pi}{d}\eta\right)\left(im\frac{\pi}{h}\right) \times$$

$$\times\, \psi_j(\eta, \tau)\, d\eta + 2\,(-1)^n \int_{-h}^{+h} \exp\left(-im\frac{\pi}{h}\xi\right) \times$$

$$\times \left(in\frac{\pi}{d}\right) \chi_j(\xi, \tau)\, d\xi\,. \tag{11}$$

Solving this system and passing to the inversions, we obtain

$$t(x, y, \tau) = A_{11} I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + 2E_1 K_{11} +$$

$$+ 2E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22}\,, \tag{12}$$

where

$$I_{kl} = \frac{1}{4hd} \iint_{(\Gamma)} \vartheta_2\left(\frac{x-\xi}{2h}\,, \frac{i\pi a_k \tau}{h^2}\right) \times$$

$$\times\, \vartheta_3\left(\frac{y-\eta}{2d}\,, \frac{i\pi a_k \tau}{d^2}\right) f_l(\xi, \eta)\, d\xi\, d\eta\,, \tag{13}$$

$$K_{kl} = \frac{1}{2hd}\left[\int_0^{\tau} d\theta \left\{\frac{\partial}{\partial x}\vartheta_3\left[\frac{h-x}{2h}, \frac{i\pi a_k(\tau-\theta)}{h^2}\right]\times\right.\right.$$

$$\times \int_{-d}^{+d} \vartheta_3\left[\frac{y-\eta}{2d}, \frac{i\pi a_k(\tau-\theta)}{d^2}\right]\psi_l(\eta,\theta)\,d\eta +$$

$$+ \frac{\partial}{\partial y}\vartheta_3\left[\frac{d-y}{2d}, \frac{i\pi a_k(\tau-\theta)}{d^2}\right]\times$$

$$\left.\left.\times \int_{-h}^{+h} \vartheta_3\left[\frac{x-\xi}{2h}, \frac{i\pi a_k(\tau-\theta)}{h^2}\right]\chi_l(\xi,\theta)\,d\xi\right\}\right]. \qquad (14)$$

The coefficients $A_i\,B_i$ are indicated in Problem 3. The function $u(x, y, \tau)$ is expressed by the right hand side of formula (12), in which $A_l\,B_i$ must be replaced by $A_i'\,B_i'$ from Problem 3.

CHAPTER 6

Heat and Mass Exchange in Molecular and Molar Transfer of Energy and Matter

In INTENSIVE heating processes, as was shown by the experimental researches of Lebedev, Maksimov, Krasnikov and others, in addition to the molecular transfer of heat and mass, there may exist in a disperse medium molar (filtrational) transfer under the action of hydrodynamical forces. Such phenomena include the movement of liquid under the influence of the expansion of trapped air, and the filtrational movement of vapour under the influence of the sustained gradient of the total pressure of the vapour–gas mixture. Molar transfer, superimposed upon the molecular processes proceeding in the medium, causes a derangement of the transfer mechanism and a consequent considerable intensification of the process. For example, in the combined drying of wood by high-frequency currents and infrared rays, an account of the excess pressure about 59 per cent of the total flux of moisture reaches the surface of the body in the form of vapour. The system of differential equations of heat and mass exchange with molecular and molar transfer of energy and mass may be presented in the following form:

$$c_q \gamma \frac{\partial t}{d\tau} = \nabla(\lambda_q \nabla t) + \varepsilon \varrho c'_m \gamma \frac{\partial \vartheta}{\partial \tau} - \sum_i c_i q_m \nabla t,$$

$$c'_m \gamma \frac{\partial \vartheta}{\partial \tau} = \nabla[\lambda_m(\nabla \theta + \delta \nabla t) + \lambda_p \nabla p], \qquad \text{(XVIII)}$$

$$c_p \gamma \frac{\partial p}{\partial \tau} = \nabla(\lambda_p \nabla p) - \varepsilon c'_m \gamma \frac{\partial \vartheta}{\partial \tau}.$$

Here λ_q is the coefficient of heat conductivity, λ_p is the coefficient of molar transfer of mass, c_p is the specific vapour capacity of the medium, p is the filtrational potential, c'_m is the specific mass capacity; the meanings of the rest of the quantities have been given above.

In order to simplify the solution of the equations it is reasonable to present them in non-dimensional forms. For a zonal calculation of high intensity heat and mass transfer for uni-dimensional bodies, these equations can have the form

$$\frac{\partial T(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 T(X, \mathrm{Fo})}{\partial X^2} - \varepsilon \mathrm{Ko} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}}, \quad \text{(XVIII')}$$

$$\frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Lu} \frac{\partial^2 \Theta(X, \mathrm{Fo})}{\partial X^2} -$$

$$- \mathrm{Lu}\,\mathrm{Pn} \frac{\partial^2 T(X, \mathrm{Fo})}{\partial X^2} - \mathrm{Lu}_p \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2},$$

$$\frac{\partial P(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Lu}_p \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2} + \varepsilon \frac{\mathrm{Ko}}{\mathrm{Bu}} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}},$$

where $\mathrm{Bu} = (\varrho c_p / C_q) \cdot p_0/(t_a - t_0)$ is the Bulygin criterion, characteristic of the storage power of the body,

$\mathrm{Lu}_a = a_p/a_q$ is the criterion of the inertial property of the pressure field in relation to the temperature field (a_p is the filtrational coefficient of potential conductivity),

$X = x/R$ is the non-dimensional coordinate,

T, Θ, P are respectively the non-dimensional potentials of transfer of heat, mass, and filtrational transfer; for a constant initial distribution of the potentials

$$T = \frac{t - t_0}{t_c - t_0}, \quad \Theta = \frac{\theta_0 - \theta}{\theta_0 - \theta_p}, \quad P = \frac{p - p_0}{p_0}.$$

§ 1. Non-stationary Potential Fields of Transfer of Energy and Matter under Boundary Conditions of the Third Kind

Let us find a non-stationary distribution of the non-dimensional potentials in a dispersed medium, approximating in form to an infinite plate, a sphere and a cylinder. We assume

problem to be symmetrical, and the initial distribution potential through the section of the material to be constant. boundary conditions can then be written in the following form:

$$\frac{\partial T(1, \mathrm{Fo})}{\partial X} - \mathrm{Bi}_q [1 - T(1, \mathrm{Fo})] + \\ + (1 - \varepsilon) \mathrm{Lu}\, \mathrm{Ko}\, \mathrm{Bi}_m [1 - \Theta(1, \mathrm{Fo})] = 0, \quad (1.1)$$

$$- \frac{\partial \Theta(1, \mathrm{Fo})}{\partial X} + \mathrm{Pn} \frac{\partial T(1, \mathrm{Fo})}{\partial X} + \\ + \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\mathrm{Lu}_p}{\mathrm{Lu}} \frac{\partial P(1, \mathrm{Fo})}{\partial X} + \mathrm{Bi}_m [1 - \Theta(1, \mathrm{Fo})] = 0, \quad (1.2)$$

$$P(1, \mathrm{Fo}) = 0, \quad (1.3)$$

$$\frac{\partial T(0, \mathrm{Fo})}{\partial X} = \frac{\partial \Theta(0, \mathrm{Fo})}{\partial X} = \frac{\partial P(0, \mathrm{Fo})}{\partial X} = 0, \quad (1.4)$$

$$T(X, 0) = 0, \ \Theta(X, 0) = 0, \ P(X, 0) = 0. \quad (1.5)$$

(a) *Infinite Plate.* Let us write

$$F_L(X, s) = \int_0^\infty F(X, \mathrm{Fo}) \exp(-s\,\mathrm{Fo})\, d\,\mathrm{Fo}. \quad (1.6)$$

Applying the integral Laplace transformation to equations (XVIII') and taking into account the initial conditions (1.5), we obtain

$$sT_L = T_L'' - \varepsilon\, \mathrm{Ko}\, s\, \Theta_L, \quad (1.7)$$

$$s\,\Theta_L = \mathrm{Lu}\, \Theta_L'' - \mathrm{Lu}\, \mathrm{Pn}\, T_L'' - \mathrm{Lu}_p \frac{\mathrm{Bu}}{\mathrm{Ko}} P_L'', \quad (1.8)$$

$$sP_L = \mathrm{Lu}_q P_L'' + \varepsilon \frac{\mathrm{Ko}}{\mathrm{Bu}} s\, \Theta_L. \quad (1.9)$$

The solution of the system of ordinary differential equations (1.7)–(1.9) in transformation can be obtained in the following way. We determine Θ_L from equation (1.7) and substitute in (1.8) and (1.9). As a result we obtain a system of two equations for T_L and P_L. The solution of the new system is found

by again using the Laplace transformation relative to the non-dimensional coordinate. The solution for Θ_L is determined by the substitution of the expressions T_L, P_L and their second derivatives in equation (1.7). After carrying out the indicated transformations we have:

$$T_L = A_1 \exp\left[\sqrt{(s)}\, \nu_1 X\right] + A_2 \exp \sqrt{(s)}\, \nu_2 X] +$$
$$+ A_3 \exp\left[\sqrt{(s)}\, \nu_3 X\right] + A_4 \exp\left[-\sqrt{(s)}\, \nu_1 X\right] +$$
$$+ A_5 \exp\left[-\sqrt{(s)}\, \nu_2 X\right] + A_6 \exp\left[-\sqrt{(s)}\, \nu_3 X\right], \quad (1.10)$$

$$\Theta_L = -\frac{1}{\varepsilon \mathrm{Ko}}\left\{A_1(1-\nu_1^2) \exp\left[\sqrt{(s)}\, \nu_1 X\right] +\right.$$
$$+ A_2(1-\nu_2^2) \exp\left[\sqrt{(s)}\, \nu_2 X\right] + A_3(1-\nu_3^2) \exp\left[\sqrt{(s)}\, \nu_3 X\right] +$$
$$+ A_4(1-\nu_1^2) \exp\left[-\sqrt{(s)}\, \nu_1 X\right] + A_5(1-\nu_2^2) \exp\left[-\sqrt{(s)}\, \nu_2 X\right] +$$
$$\left. + A_6(1-\nu_3^2) \exp\left[-\sqrt{(s)}\, \nu_3 X\right]\right\}, \quad (1.11)$$

$$P_L = \frac{1}{\varepsilon \mathrm{Bu}}\left\{A_1 \sigma_1 \exp\left[\sqrt{(s)}\, \nu_1 X\right] + A_2 \sigma_2 \exp\left[\sqrt{(s)}\, \nu_2 X\right] +\right.$$
$$+ A_3 \sigma_3 \exp\left[\sqrt{(s)}\, \nu_3 X\right] + A_4 \sigma_1 \exp\left[-\sqrt{(s)}\, \nu_1 X\right] +$$
$$\left. + A_5 \sigma_2 \exp\left[-\sqrt{(s)}\, \nu_2 X\right] + A_6 \sigma_3 \exp\left[-\sqrt{(s)}\, \nu_3 X\right]\right\}. \quad (1.12)$$

Here

$$\sigma_j = (1-\varepsilon)(1-\nu_j^2) - \mathrm{Lu}(1-\nu_j^2)\nu_j^2 - \varepsilon\, \mathrm{Ko}\, \mathrm{Pn}\, \mathrm{Lu}\, \nu_j^2, \quad (1.13)$$

$$\nu_j = \sqrt{\left(\Lambda_j + \frac{1}{3}\alpha\right)}, \quad (j = 1, 2, 3). \quad (1.14)$$

Λ_j, which enters into (1.14), is determined by the expression

$$\Lambda = \sqrt[3]{\left[-\frac{1}{2}\Pi_2 + \sqrt{\left(\frac{1}{4}\Pi_2^2 + \frac{1}{27}\Pi_1^3\right)}\right]} +$$
$$+ \sqrt[3]{\left[-\frac{1}{2}\Pi_2 - \sqrt{\left(\frac{1}{4}\Pi_2^2 + \frac{1}{27}\Pi_1^3\right)}\right]}.$$

Here the correct values of A must simultaneously satisfy the relation

$$\sqrt[3]{\left[-\frac{1}{2}\Pi_2+\sqrt{\left(\frac{1}{4}\Pi_2^2+\frac{1}{27}\Pi_1^3\right)}\right]}\times$$

$$\times\sqrt[3]{\left[-\frac{1}{2}\Pi_2-\sqrt{\left(\frac{1}{4}\Pi_2^2+\frac{1}{27}\Pi_1^3\right)}\right]}=-\frac{1}{3}\Pi_1,$$

where

$$\Pi_1=-\frac{a^2}{3}+\beta,\quad \Pi_2=-\frac{2a^3}{27}+\frac{a\beta}{3}-\gamma,$$

$$a=1+(1-\varepsilon)\frac{1}{\mathrm{Lu}}+\frac{1}{\mathrm{Lu}_p}+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn},$$

$$\beta=(1-\varepsilon)\frac{1}{\mathrm{Lu}}+\left(1+\frac{1}{\mathrm{Lu}}+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\right)\frac{1}{\mathrm{Lu}_p},$$

$$\gamma=\frac{1}{\mathrm{Lu}\,\mathrm{Lu}_p}.$$

The values of ν_j for different values of the criteria of heat and mass exchange are shown in Table 36.

Thus, the general solution of the system of equations (XVIII′) in Laplace transformation is given by the expressions (1.10)–(1.12) where $A_K(K=1,\ 2,\ \ldots,\ 6)$ are constant coefficients with respect to X. The values of these we found from the limiting conditions (1.1)–(1.4). We first make use of the symmetry condition (1.4). After the application of the Laplace transformation (1.6) they can be written thus:

$$T'_L(0,s)=\Theta'_L(0,s)=P'_L(0,s)=0.$$

Since the general solutions must satisfy the symmetry conditions, we have

$$A_1=A_4,\quad A_2=A_5,\quad A_3=A_6.$$

On the basis of these relations the solutions of (1.10) – (1.12) can be rewritten in the following form:

$$T_L(X,s)=\sum_{j=1}^{3}C_j\cosh\sqrt{(s)}\,\nu_j X, \tag{1.15}$$

$$\Theta_L(X,s)=-\frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{j=1}^{3}C_j(1-\nu_j^2)\cosh\sqrt{(s)}\,\nu_j X, \tag{1.16}$$

$$P_2(X, s) = \frac{1}{\varepsilon \,\mathrm{Bu}} \sum_{j=1}^{3} C_j \sigma_j \cosh \sqrt{(s)}\, \nu_j X\,, \tag{1.17}$$

where $C_j (j = 1, 2, 3)$ are new constants relative to the coordinates. Applying transformation (1.6) to the boundary conditions (1.1), (1.2), we obtain

$$T'_L(1, s) - \mathrm{Bi}_q \left[\frac{1}{s} - T_L(1, s)\right] +$$

$$+ (1 - \varepsilon)\,\mathrm{Lu}\,\mathrm{Ko}\,\mathrm{Bi}_m \left[\frac{1}{s} - \Theta_L(1, s)\right] = 0\,, \tag{1.18}$$

$$- \Theta'_L(1, s) + \mathrm{Pn}\, T'_E(1, s) +$$

$$+ \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\mathrm{Lu}_p}{\mathrm{Lu}} P'_L(1, s) + \mathrm{Bi}_m \left[\frac{1}{s} - \Theta_L(1, s)\right] = 0\,, \tag{1.19}$$

$$P_L(1, s) = 0\,. \tag{1.20}$$

Substituting in conditions (1.18)–(1.20) the solutions (1.15)–(1.17) we obtain the system of algebraic equations

$$\left.\begin{aligned} &\sum_{j=1}^{3} C_j P_j + \frac{\varepsilon \,\mathrm{Ko}}{s} = 0 \\ &\sum_{j=1}^{3} C_j Q_j + \frac{\varepsilon \,\mathrm{Ko}\, K_1 - 1}{s} = 0 \\ &\sum_{j=1}^{3} C_j \chi_j = 0 \end{aligned}\right\}, \tag{1.21}$$

the solution of which will give us

$$C_1 = \frac{(\varepsilon \,\mathrm{Ko}\, K_1 - 1)\left(P_2 - \frac{\chi_2}{\chi_3} P_3\right) - \varepsilon \mathrm{Ko}\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_2} Q_3\right)\right]}, \tag{1.22}$$

$$C_2 = -\frac{(\varepsilon \,\mathrm{Ko}\, K_1 - 1)\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right) - \varepsilon \,\mathrm{Ko}\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]}, \tag{1.23}$$

$$C_3 = \frac{(\varepsilon \mathrm{Ko}\, K_1 - 1)\left(P_1 \frac{\chi_2}{\chi_3} - P_2 \frac{\chi_1}{\chi_3}\right) - \varepsilon \mathrm{Ko}\left(Q_1 \frac{\chi_2}{\chi_3} - Q_2 \frac{\chi_1}{\chi_3}\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]}, \quad (1.24)$$

where

$$P_j = (1 - \nu_j^2) \cosh \sqrt{(s)}\, \nu_j + \left[\frac{(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\, \mathrm{Pn} + \frac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m}\right] \sqrt{(s)}\, \nu_j \sinh \sqrt{(s)}\, \nu_j,$$

$$Q_j = [1 + (1 - \nu_j^2) K_1] \cosh \sqrt{(s)}\, \nu_j + \frac{1}{\mathrm{Bi}_p} \sqrt{(s)}\, \nu_j \sinh \sqrt{(s)}\, \nu_j,$$

$$\chi_j = \sigma_j \cosh \sqrt{(s)}\, \nu_j.$$

To obtain the inversions of the transform, each of the solutions must be presented in the form of the ratio of two generalized polynomials in s: $\Phi(s)/\psi(s)$. The denominator of the corresponding ratios for T_L, Θ_L, P_L will be the same and equal to

$$\psi(s) = s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]. \quad (1.25)$$

An analysis of the ratios and the roots of the denominator (1.25) shows that the conditions essential for transition to the original, in accordance with the expansion theorem, are fulfilled, and we can make use of the following formula of inverse transformation:

$$L^{-1}\left[\frac{\Phi(s)}{\psi(s)}\right] = \frac{\Phi(0)}{\psi'(0)} + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)} \exp(s_n \mathrm{Fo}), \quad (1.26)$$

where $s = 0$ and $s = s_n$ are the roots of (1.25).

Determining $\psi'(0)$, $\Phi(0)$, $\psi'(s_n)$, $\Phi(s_n)$ for the fields of non-dimensional transfer potentials and substituting the values found in (1.26), after a number of transformations we finally obtain the solution of the problem under consideration:

$$T(X, \mathrm{Fo}) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \cos \nu_j \mu_n X \exp(-\mu_n^2 \mathrm{Fo}), \quad (1.27)$$

$$\Theta(X, \mathrm{Fo}) = 1 + \frac{1}{\varepsilon\,\mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \times$$

$$\times C_{nj}(1 - \nu_j^2) \cos \nu_j \mu_n X \exp(-\mu_n^2 \mathrm{Fo}), \quad (1.28)$$

$$P(X, \mathrm{Fo}) = -\frac{1}{\varepsilon\,\mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \times$$

$$\times C_{nj}\sigma_j \cos \nu_j \mu_n X \exp(-\mu_n^2 \mathrm{Fo}). \quad (1.29)$$

The coefficients C_{nj} are determined by the expressions:

$$C_{n1} = 2(\varepsilon\,\mathrm{Ko}\,K_1 - 1)\left(P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3}\right) -$$

$$- \varepsilon\,\mathrm{Ko}\left(Q_{n2} - \frac{\chi_{n3}}{\chi_{n3}} Q_{n3}\right)(\mu_n \psi_n)^{-1}, \quad (1.30)$$

$$C_{n2} = -2(\varepsilon\,\mathrm{Ko}\,K_1 - 1)\left(P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3}\right) -$$

$$- \varepsilon\,\mathrm{Ko}\left(Q_{n1} - \frac{\chi_{n1}}{\chi_{n2}} Q_{n3}\right)(\mu_n \psi_n)^{-1}, \quad (1.31)$$

$$C_{n3} = 2(\varepsilon\,\mathrm{Ko}\,K_1 - 1)\left(P_{n1}\frac{\chi_{n2}}{\chi_{n3}} - P_{n2}\frac{\chi_{n1}}{\chi_{n3}}\right) -$$

$$- \varepsilon\,\mathrm{Ko}\left(Q_{n1}\frac{\chi_{n2}}{\chi_{n3}} - Q_{n2}\frac{\chi_{n1}}{\chi_{n2}}\right)(\mu_n \psi_n)^{-1}, \quad (1.32)$$

where

$$\psi_n = \left(P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3}\right)\left[\nu_2 A_{n2} - \frac{\chi_{n2}}{\chi_{n3}}(\nu_3 A_{n3} + b_{n2} Q_{n3})\right] +$$

$$+ \left[\nu_1 B_{n1} - \frac{\chi_{n1}}{\chi_{n3}}(\nu_3 B_{n3} + b_{n1} P_{n3})\right]\left(Q_{n2} - \frac{\chi_{n2}}{\chi_{n2}} Q_{n3}\right) -$$

$$- \left(P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3}\right)\left[\nu_1 A_{n1} - \frac{\chi_{n1}}{\chi_{n3}}(\nu_3 A_{n3} + b_{n1} Q_{n3})\right] -$$

$$- \left[\nu_2 B_{n2} - \frac{\chi_{n2}}{\chi_{n3}}(\nu_3 B_{n3} + b_{n2} P_{n3})\right]\left(Q_{n1} - \frac{\chi_{n1}}{\chi_{n3}} Q_{n3}\right), \quad (1.33)$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n, \quad (1.34)$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n + \frac{(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} \times$$

$$\times (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n), \quad (1.35)$$

$$b_{nj} = \nu_j \tan \nu_j \mu_n - \nu_3 \tan \nu_3 \mu_n , \tag{1.36}$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n - \frac{(1 - \nu_j^2) + \varepsilon \, \mathrm{Ko} \, \mathrm{Pn} + \frac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} \times$$

$$\times \nu_j \mu_n \sin \nu_j \mu_n , \tag{1.37}$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n , \tag{1.38}$$

$$\chi_{nj} = \sigma_j \cos \nu_j \mu_n . \tag{1.39}$$

The characteristic roots $\mu_n = i \sqrt{s_n}$ are determined by the solution of the following equation

$$\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) -$$

$$- \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right) = 0 . \tag{1.40}$$

In Tables 37–41 and Figs. 48–52 the values of the characteristic roots μ_n for different criteria of similarity as they depend on Bi_q are given. The values of μ_n are obtained by the graphic-analytic solution of equation (1.40).

The mean value of a non-dimensional quantity is determined by the formula

$$\overline{\varphi}(\mathrm{Fo}) = \int_0^1 \varphi(X, \mathrm{Fo}) \, dX .$$

Hence, for the mean non-dimensional potentials of transfer we have

$$\overline{T} = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} \exp(-\mu_n^2 \mathrm{Fo}) , \tag{1.41}$$

$$\overline{\Theta} = 1 + \frac{1}{\varepsilon \, \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} (1 - \nu_j^2) \exp(-\mu_n^2 \mathrm{Fo}) , \tag{1.42}$$

$$\overline{P} = -\frac{1}{\varepsilon \, \mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} \sigma_j \exp(-\mu_n^2 \mathrm{Fo}) . \tag{1.43}$$

Here

$$D_{nj} = C_{nj} \frac{\sin \nu_j \mu_n}{\nu_j \mu_n} .$$

TABLE 37. ROOTS OF THE CHARACTERISTIC EQUATION (1.40) IN RELATION TO Bi_m (Lu = 0·3, ε = 0·7, KoPn = 2·25, Lu_p = 500)

Bi_q	$Bi_m = 5$			$Bi_m = 10$			$Bi_m = 15$			$Bi_m = 20$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0·30	0·454	1·221	1·702	0·466	—	—	0·468	—	—	0·469	—	—
0·35	0·476	1·358	1·700	0·492	—	—	0·494	—	—	0·495	—	—
0·40	0·493	1·392	1·697	0·513	—	—	0·515	—	—	0·517	—	—
0·45	0·511	1·429	1·686	0·532	—	—	0·534	—	—	0·536	—	—
0·50	0·528	1·462	1·668	0·552	—	—	0·553	—	—	0·556	—	—
0·60	0·555	—	—	0·577	—	—	0·581	—	—	0·583	—	—
0·70	0·578	—	—	0·599	—	—	0·605	—	—	0·607	—	—
0·80	0·596	—	—	0·617	—	—	0·624	—	—	0·627	—	—
0·90	0·611	—	—	0·633	—	—	0·640	—	—	0·645	—	—
1·00	0·622	—	—	0·645	—	—	0·653	—	—	0·658	—	—
1·50	0·656	—	—	0·686	—	—	0·700	—	—	0·706	—	—
2·00	0·676	—	—	0·714	—	—	0·725	—	—	0·733	—	—
3·00	0·696	—	—	0·739	—	—	0·755	—	—	0·764	—	—
4·00	0·704	—	—	0·752	—	—	0·771	—	—	0·779	—	—
5·00	0·712	1·990	2·091	0·761	—	—	0·781	—	—	0·789	—	—
6·00	0·715	1·931	2·211	0·767	—	—	0·787	—	—	0·796	—	—
7·00	0·718	1·911	2·286	0·771	—	—	0·792	—	—	0·801	—	—
8·00	0·721	1·900	2·336	0·775	—	—	0·796	—	—	0·804	—	—
9·00	0·723	1·895	2·370	0·777	—	—	0·798	—	—	0·807	—	—
10·00	0·725	1·891	2·397	0·779	—	—	0·801	—	—	0·810	—	—
15·00	0·728	1·879	2·483	0·786	2·213	2·462	0·808	—	—	0·817	—	—
20·00	0·730	1·875	2·529	0·788	2·185	2·527	0·812	2·378	2·506	0·820	—	—
30·00	0·732	1·870	2·579	0·792	2·158	2·585	0·816	2·326	2·586	0·824	2·424	2·587
40·00	0·734	1·867	2·600	0·794	2·149	2·610	0·817	2·311	2·610	0·826	2·404	2·610
50·00	0·735	1·866	2·620	0·795	2·144	2·630	0·818	2·302	2·630	0·827	2·392	2·630

TABLE 38. ROOTS OF THE CHARACTERISTIC EQUATION (1.40) IN RELATION TO Pn ($Lu = 0{\cdot}3$, $\varepsilon = 0{\cdot}7$, $Bi_m = 20$, $Ko = 0$, $Lu_p = 500$)

Bi_f	$Pn = 0{\cdot}65$		$Pn = 0{\cdot}45$		$Pn = 0{\cdot}25$			$Pn = 0{\cdot}05$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_3	μ_1	μ_2
0·30	0·412	1·752	0·443	2·480	0·460	—	—	0·508	1·437
0·35	0·430	1·761	0·464	2·464	0·496	—	—	0·544	1·430
0·40	0·445	1·771	0·482	2·430	0·517	—	—	0·574	1·445
0·45	0·458	1·779	0·498	2·402	0·530	—	—	0·602	1·450
0·50	0·470	1·784	0·511	2·375	0·550	—	—	0·625	1·453
0·60	0·488	1·792	0·534	2·330	0·583	—	—	0·669	1·466
0·70	0·504	1·799	0·552	2·288	0·607	—	—	0·700	1·477
0·80	0·515	1·804	0·566	2·260	0·627	—	—	0·741	1·488
0·90	0·525	1·810	0·578	2·237	0·646	—	—	0·771	1·498
1·00	0·533	1·815	0·588	2·220	0·658	—	—	0·796	1·507
1·50	0·558	1·830	0·622	2·177	0·706	—	—	0·885	1·550
2·00	0·572	1·837	0·642	2·159	0·733	—	—	0·938	1·507
3·00	0·580	1·847	0·662	2·142	0·764	—	—	0·993	1·600
4·00	0·599	1·854	0·673	2·135	0·770	—	—	1·023	1·701
5·00	0·604	1·858	0·681	2·131	0·789	—	—	1·042	1·734
6·00	0·608	1·861	0·685	2·128	0·796	—	—	1·054	1·755
7·00	0·611	1·863	0·688	2·126	0·801	—	—	1·061	1·773
8·00	0·613	1·865	0·691	2·124	0·804	—	—	1·068	1·787
9·00	0·614	1·866	0·693	2·123	0·807	—	—	1·073	1·797
10·00	0·616	1·869	0·695	2·122	0·810	—	—	1·077	1·806
15·00	0·621	1·870	0·699	2·120	0·817	—	—	1·090	1·831
20·00	0·622	1·872	0·701	2·118	0·820	—	—	1·096	1·846
30·00	0·624	1·873	0·704	2·117	2·824	2·424	2·587	1·103	1·862
40·00	0·626	1·874	0·705	2·116	0·826	2·404	2·610	1·105	1·870
50·00	0·627	1·875	0·706	2·115	0·827	2·392	2·630	1·106	1·875

TABLE 39. ROOTS OF THE CHARACTERISTIC EQUATION (1.40) IN RELATION TO Ko (Lu = 0·3, $\varepsilon = 0{\cdot}7$, $\mathrm{Bi}_m = 20$, Pn = 0·25, $\mathrm{Lu}_q = 500$)

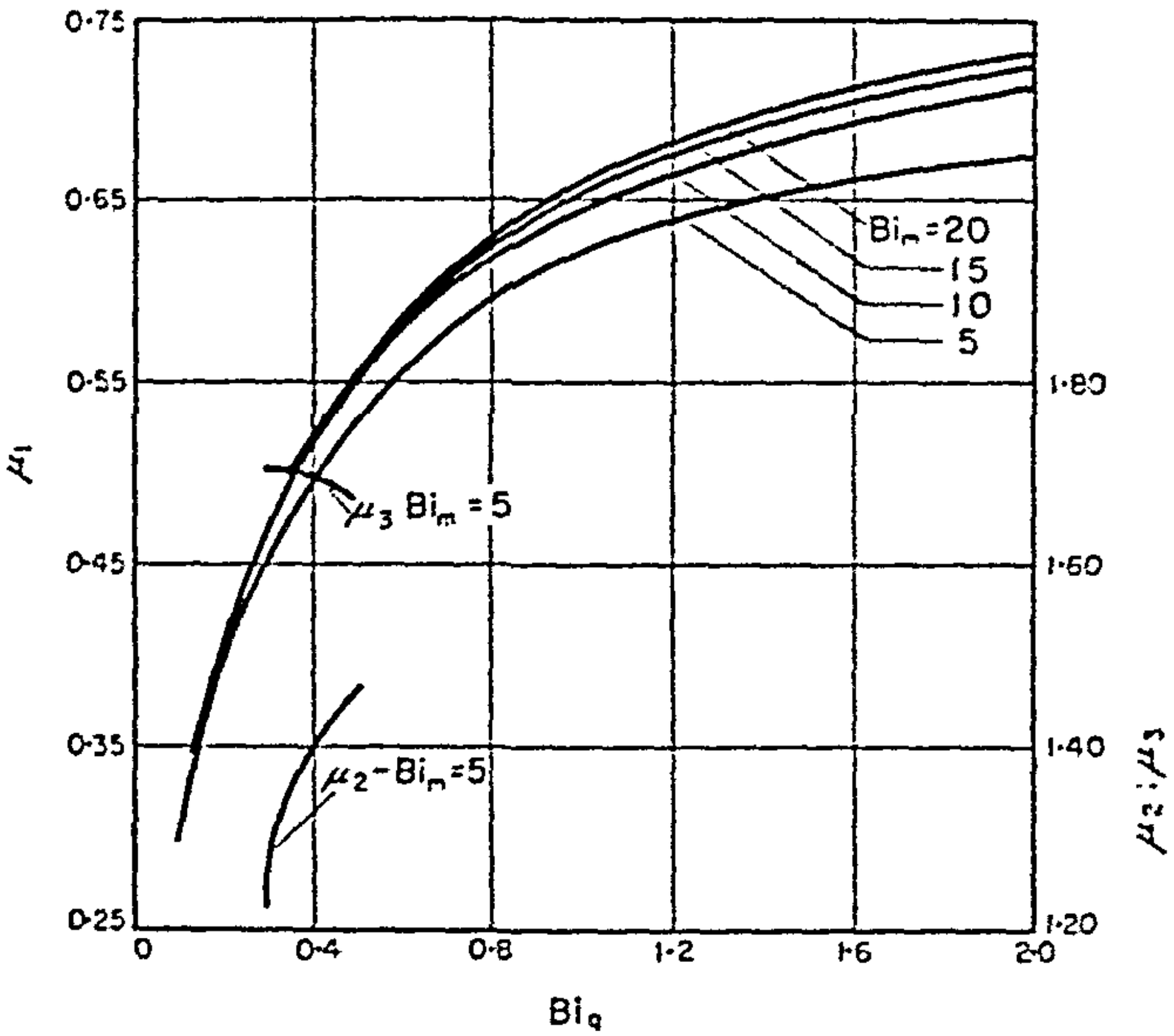
Bi_q	Ko = 3		Ko = 6			Ko = 9			Ko = 12		
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0·30	0·502	1·489	0·485	1·686	2·191	0·469	—	—	0·456	—	—
0·35	0·535	1·498	0·513	1·699	2·189	0·495	—	—	0·480	—	—
0·40	0·565	1·508	0·538	1·720	2·187	0·517	—	—	0·501	—	—
0·45	0·590	1·518	0·563	1·743	2·185	0·536	—	—	0·518	—	—
0·50	0·614	1·527	0·582	1·768	2·182	0·556	—	—	0·534	—	—
0·60	0·656	1·546	0·617	1·816	2·179	0·583	—	—	0·560	—	—
0·70	0·690	1·563	0·646	1·862	2·173	0·607	—	—	0·581	—	—
0·80	0·718	1·579	0·670	1·904	2·163	0·627	—	—	0·599	—	—
0·90	0·743	1·595	0·690	1·948	2·148	0·645	—	—	0·613	—	—
1·00	0·766	1·612	0·707	1·988	2·129	0·658	—	—	0·625	—	—
1·50	0·843	1·687	0·765	—	—	0·706	—	—	0·666	—	—
2·00	0·886	1·734	0·798	—	—	0·733	—	—	0·690	—	—
3·00	0·934	1·807	0·836	—	—	0·764	—	—	0·716	—	—
4·00	0·961	1·852	0·856	—	—	0·779	—	—	0·730	—	—
5·00	0·975	1·884	0·867	—	—	0·789	—	—	0·738	—	—
6·00	0·987	1·907	0·875	2·340	2·422	0·796	—	—	0·744	—	—
7·00	0·995	1·924	0·881	2·317	2·503	0·801	—	—	0·748	—	—
8·00	1·000	1·937	4·886	2·307	2·547	0·804	—	—	0·751	2·415	2·575
9·00	1·005	1·946	0·889	2·302	2·569	0·807	—	—	0·754	2·386	2·603
10·00	1·008	1·956	0·891	2·296	2·605	0·810	—	—	0·756	2·358	2·639
15·00	1·020	1·978	0·900	2·287	2·671	0·817	—	—	0·761	2·328	2·700
20·00	1·025	1·991	0·904	2·282	2·710	0·820	—	—	0·764	2·315	2·725
30·00	1·030	2·003	0·908	2·270	2·747	0·824	2·424	2·587	0·766	2·303	2·747
40·00	1·033	2·007	0·911	2·277	2·768	0·826	2·404	2·610	0·769	2·299	2·759
50·00	1·035	2·012	0·913	2·276	2·777	0·827	2·392	2·630	0·770	2·295	2·764

TABLE 40. ROOTS OF THE CHARACTERISTIC EQUATION (1.40) IN RELATION TO Lu_p ($Lu = 0{\cdot}3$, $\varepsilon = 0{\cdot}7$, $Bi_m = 20$, $KoPn = 2{\cdot}25$)

Bi_q	$Lu_p = 300$			$Lu_p = 500$			$Lu_p = 1000$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0·30	0·454	1·677	2·260	0·469	—	—	0·468	—	—
0·35	0·480	1·700	2·246	0·495	—	—	0·493	—	—
0·40	0·503	1·726	2·238	0·517	—	—	0·517	—	—
0·45	0·523	1·754	2·232	0·536	—	—	0·537	—	—
0·50	0·541	1·778	2·228	0·556	—	—	0·554	—	—
0·60	0·572	1·836	2·222	0·583	—	—	0·584	—	—
0·70	0·596	1·889	2·214	0·607	—	—	0·607	—	—
0·80	0·617	1·942	2·199	0·627	—	—	0·628	—	—
0·90	0·634	2·005	2·173	0·645	—	—	0·644	—	—
1·00	0·648	—	—	0·658	—	—	0·658	—	—
1·50	0·699	—	—	0·706	—	—	0·707	—	—
2·00	0·727	—	—	0·733	—	—	0·734	—	—
3·00	0·758	—	—	0·764	—	—	0·765	—	—
4·00	0·775	—	—	0·779	—	—	0·781	—	—
5·00	0·786	—	—	0·789	—	—	0·791	—	—
6·00	0·793	—	—	0·796	—	—	0·799	—	—
7·00	0·798	—	—	0·801	—	—	0·803	—	—
8·00	0·802	—	—	0·804	—	—	0·807	—	—
9·00	0·805	—	—	0·807	—	—	0·810	—	—
10·00	0·808	—	—	0·810	—	—	0·813	—	—
15·00	0·816	—	—	0·817	—	—	0·820	—	—
20·00	0·819	—	—	0·820	—	—	0·824	—	—
30·00	0·823	—	—	0·824	2·424	2·587	0·828	2·428	2·581
40·00	0·825	—	—	0·826	2·404	2·610	0·830	2·404	2·616
50·00	0·826	—	—	0·827	2·392	2·630	0·831	2·394	2·635

TABLE 41. RELATION OF THE CHARACTERISTIC ROOTS OF EQUATION (1.40) TO ϵ. (Lu = 0·3, Lu_p = 500, Bi_m = 20, Pn = 0·25; Ko = 0)

Bi_q	ϵ = 0·1			ϵ = 0·7			ϵ = 0·9	
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2
0·30	0·463	1·345	1·950	0·460	—	—	0·474	—
0·35	0·490	1·351	1·961	0·495	—	—	0·501	—
0·40	0·512	1·358	1·967	0·517	—	—	0·523	—
0·45	0·533	1·360	1·970	0·536	—	—	0·543	—
0·50	0·550	1·381	1·978	0·556	—	—	0·561	—
0·60	0·579	1·405	1·993	0·583	—	—	0·590	—
0·70	0·603	1·422	2·012	0·607	—	—	0·614	—
0·80	0·623	1·438	2·030	0·627	—	—	0·633	—
0·90	0·639	1·453	2·047	0·645	—	—	0·649	—
1·00	0·653	1·466	2·062	0·658	—	—	0·662	—
1·50	0·702	1·510	2·123	0·706	—	—	0·700	—
2·00	0·731	1·530	2·178	0·733	—	—	0·730	—
3·00	0·701	1·568	2·247	0·764	—	—	0·767	—
4·00	0·770	1·586	2·286	0·779	—	—	0·784	2·016
5·00	0·785	1·598	2·313	0·780	—	—	0·794	2·560
6·00	0·702	1·607	2·334	0·706	—	—	0·803	2·537
7·00	0·706	1·613	2·352	0·801	—	—	0·807	2·523
8·00	0·800	1·617	2·364	0·804	—	—	0·812	2·514
9·00	0·802	1·620	2·376	0·807	—	—	0·815	2·507
10·00	0·804	1·623	2·384	0·810	—	—	0·817	2·502
15·00	0·810	1·631	2·413	0·817	—	—	0·825	2·490
20·00	0·813	1·636	2·427	0·820	—	—	0·828	2·484
30·00	0·816	1·640	2·442	0·824	2·424	2·587	0·832	2·478
40·00	0·818	1·642	2·450	0·826	2·404	2·610	0·834	2·475
50·00	0·819	1·644	2·455	0·827	2·302	2·630	0·836	2·473



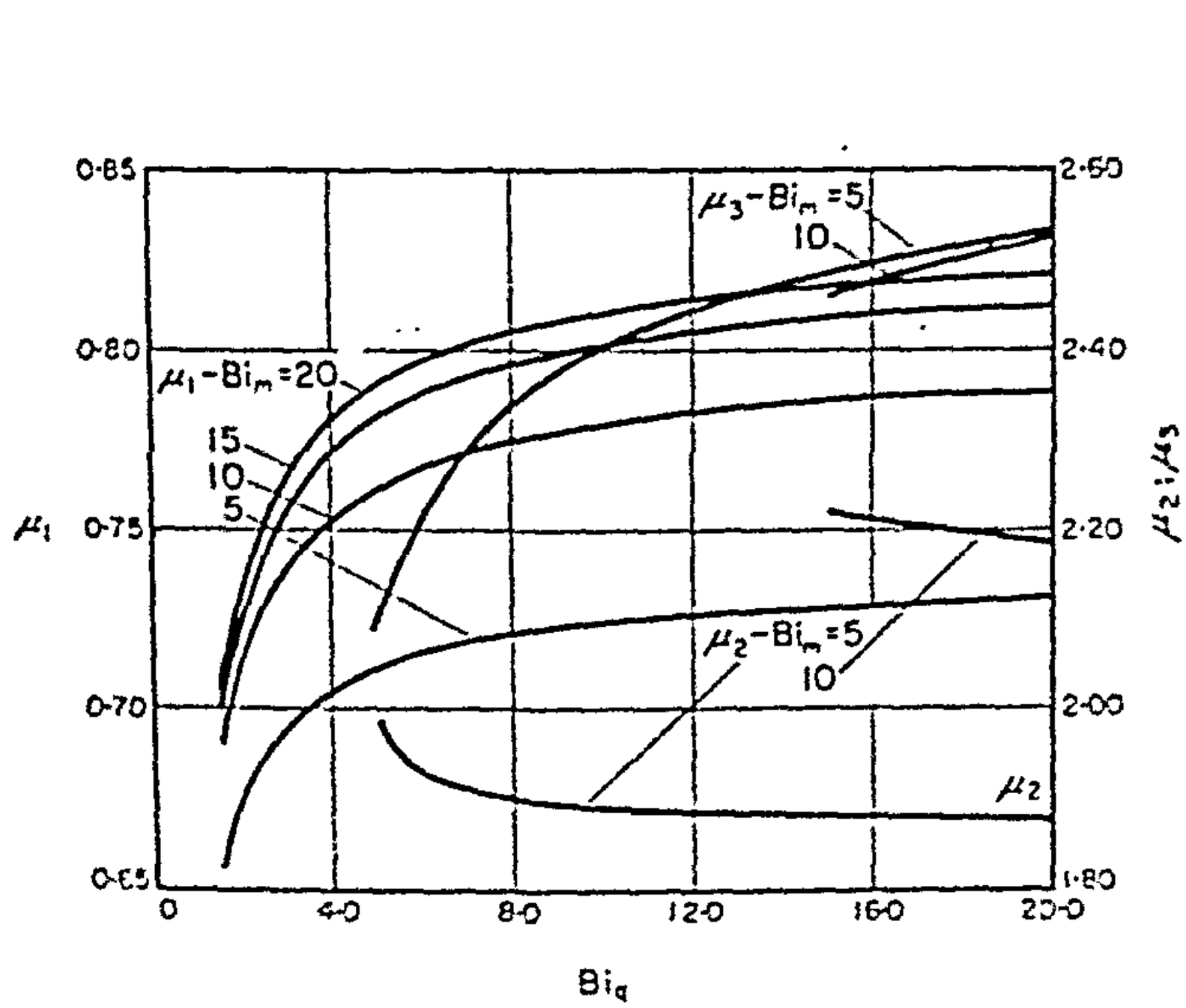


Fig. 48. Relation of the characteristic roots for an infinite plate: (*a*) — to Bi_m ($Bi_q = 0{\cdot}1$ to $2{\cdot}0$); (*b*) — to Bi_m($Bi_q = 1{\cdot}0$ to 20)

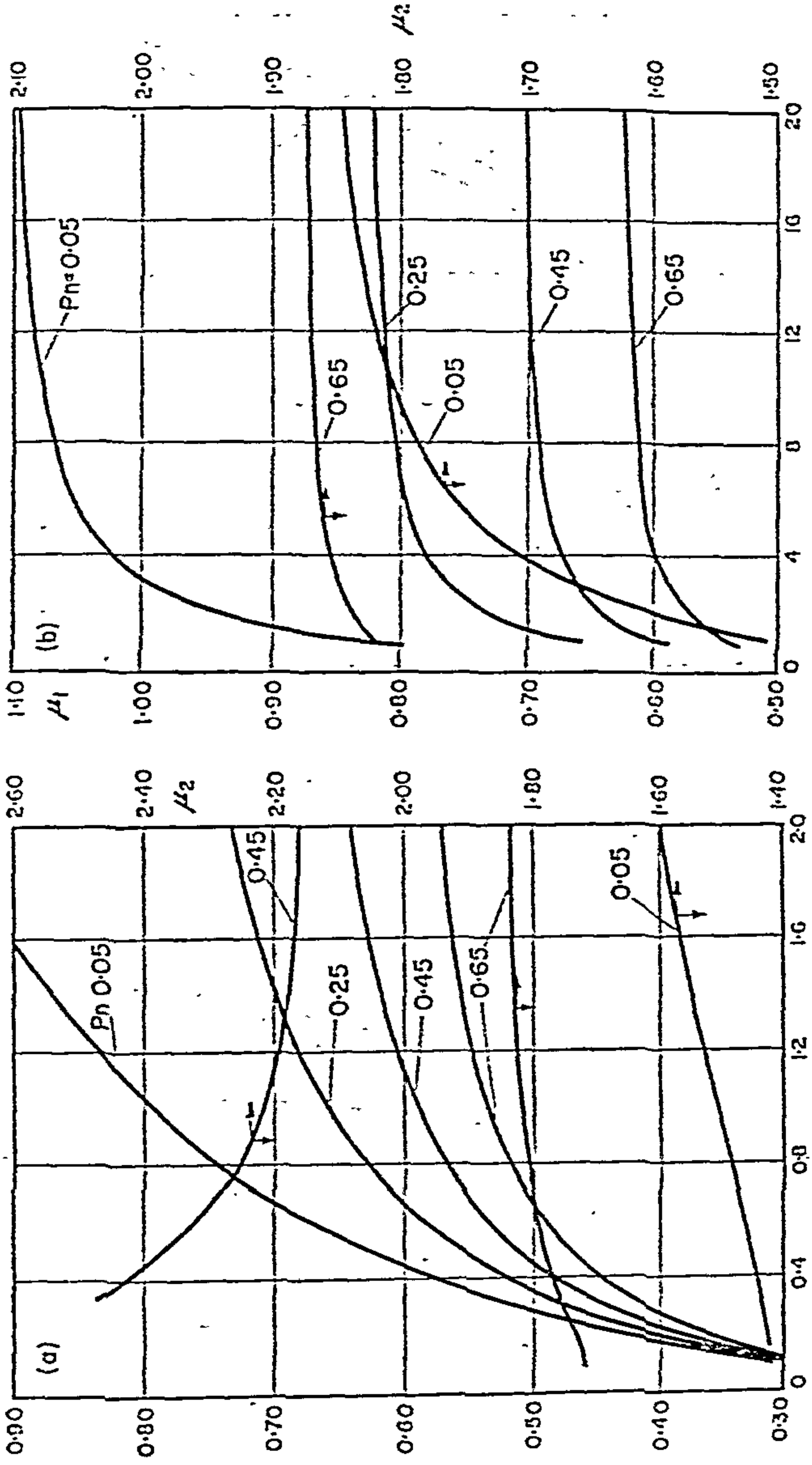


FIG. 49. Relation of the dimensionless temperatures for an infinite plate: (*a*) — to Pn ($Bi_q = 1.0$ to 20); (*b*) — to Pn ($Bi_q = 0.1$ to 2.0); (*b*) — to Bi_q (Pn = 0.05 to 0.65)

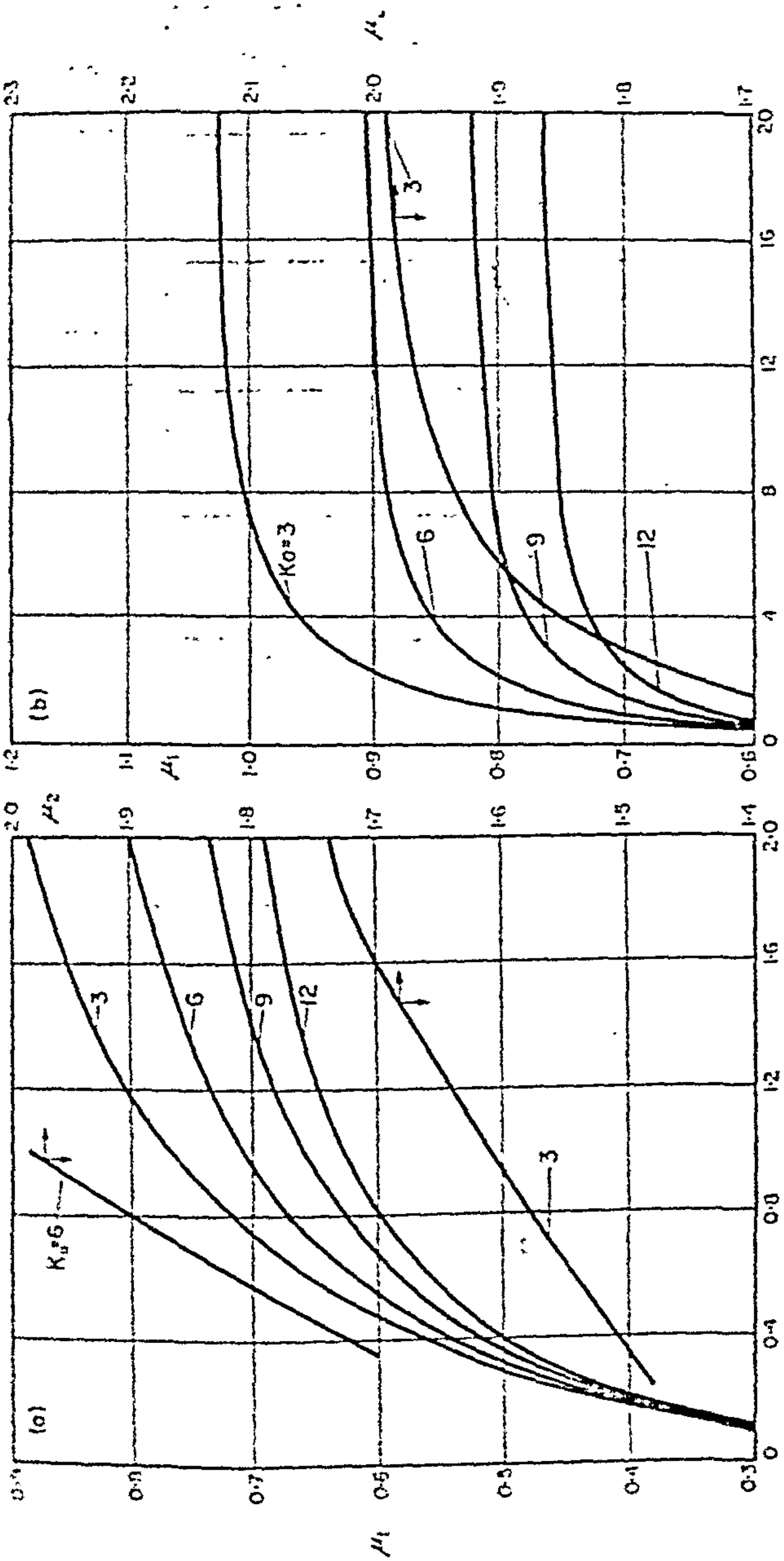
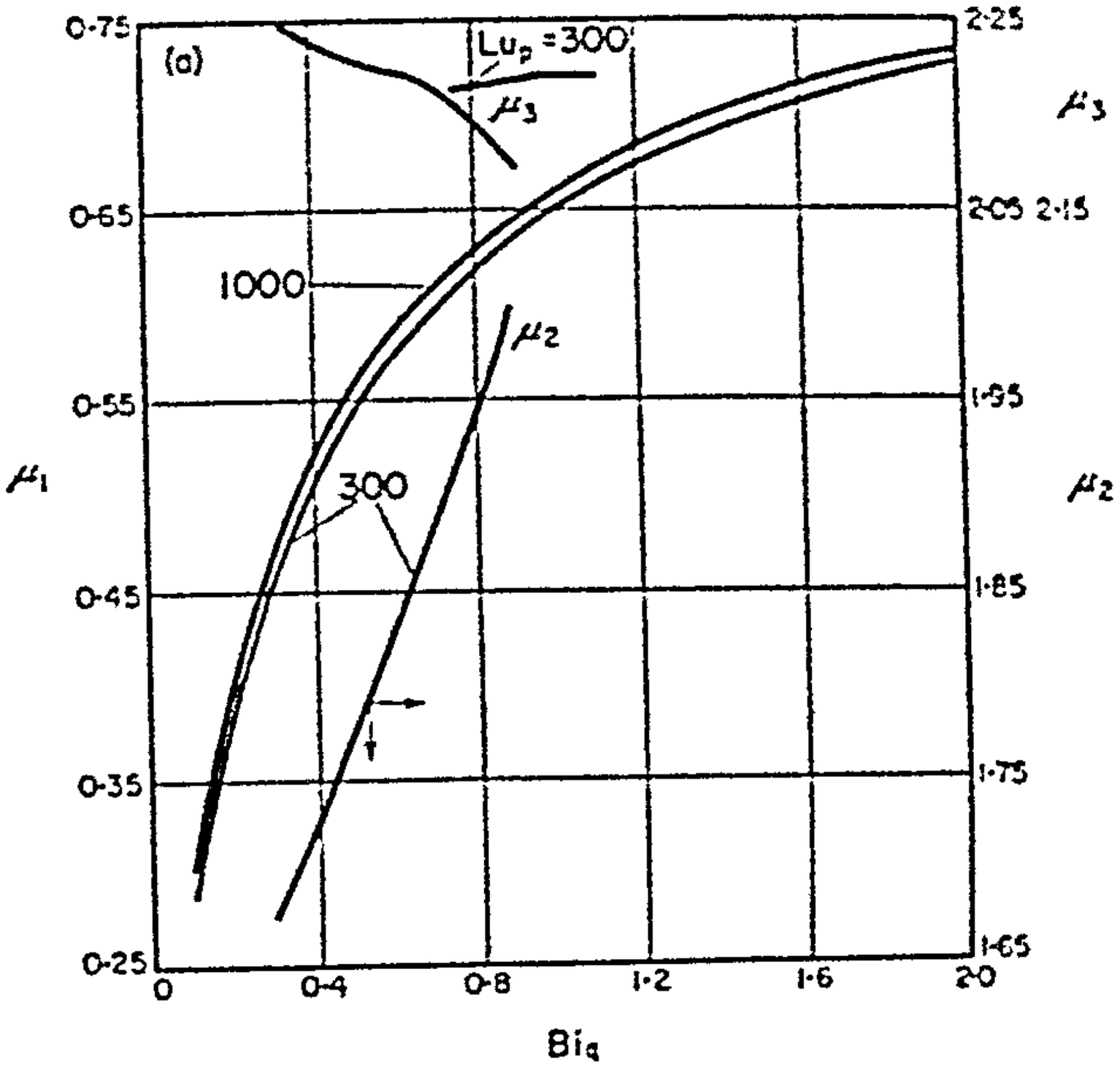


Fig. 50. Relation of the characteristic roots for an infinite plate: (*a*) — to Ko (Bi_q = 1·0 to 20), (*b*) — to Bi_q (Ko = 0·1 to 2·0), (*b*) — to Bi_q (Ko = 1·0 to 20)



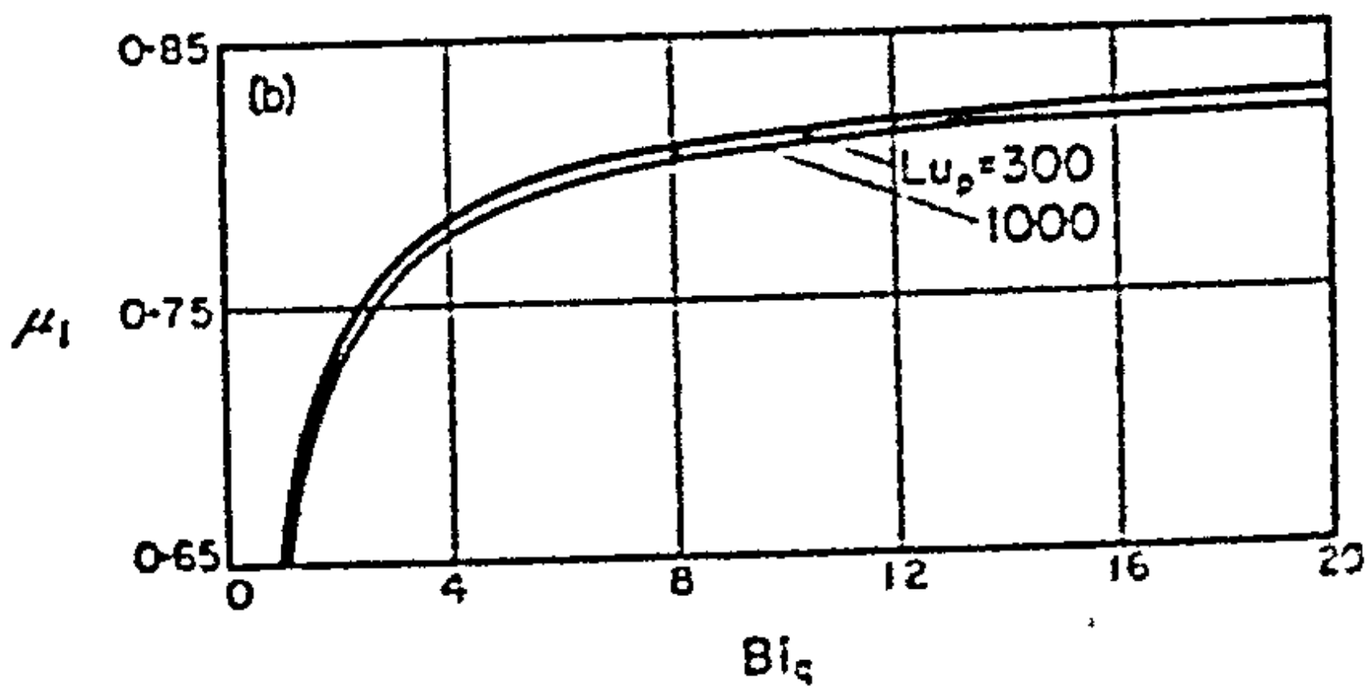


FIG. 51. Relation of the characteristic roots for an infinite plate: (*a*) — to Lu_p ($Bi_q = 0{\cdot}1$ to $2{\cdot}0$); (*b*) — to Lu_p ($Bi_q = 1{\cdot}0$ to 20)

The expressions for the rates of change of the corresponding parameters, as well as for the expenditure of heat and of matter in the process of heat and mass exchange, are easy to obtain by making use of the known formulae and solutions given above. The results of numerical calculations by the formulae (1.27)–(1.29), (1.41)–(1.43) show that the

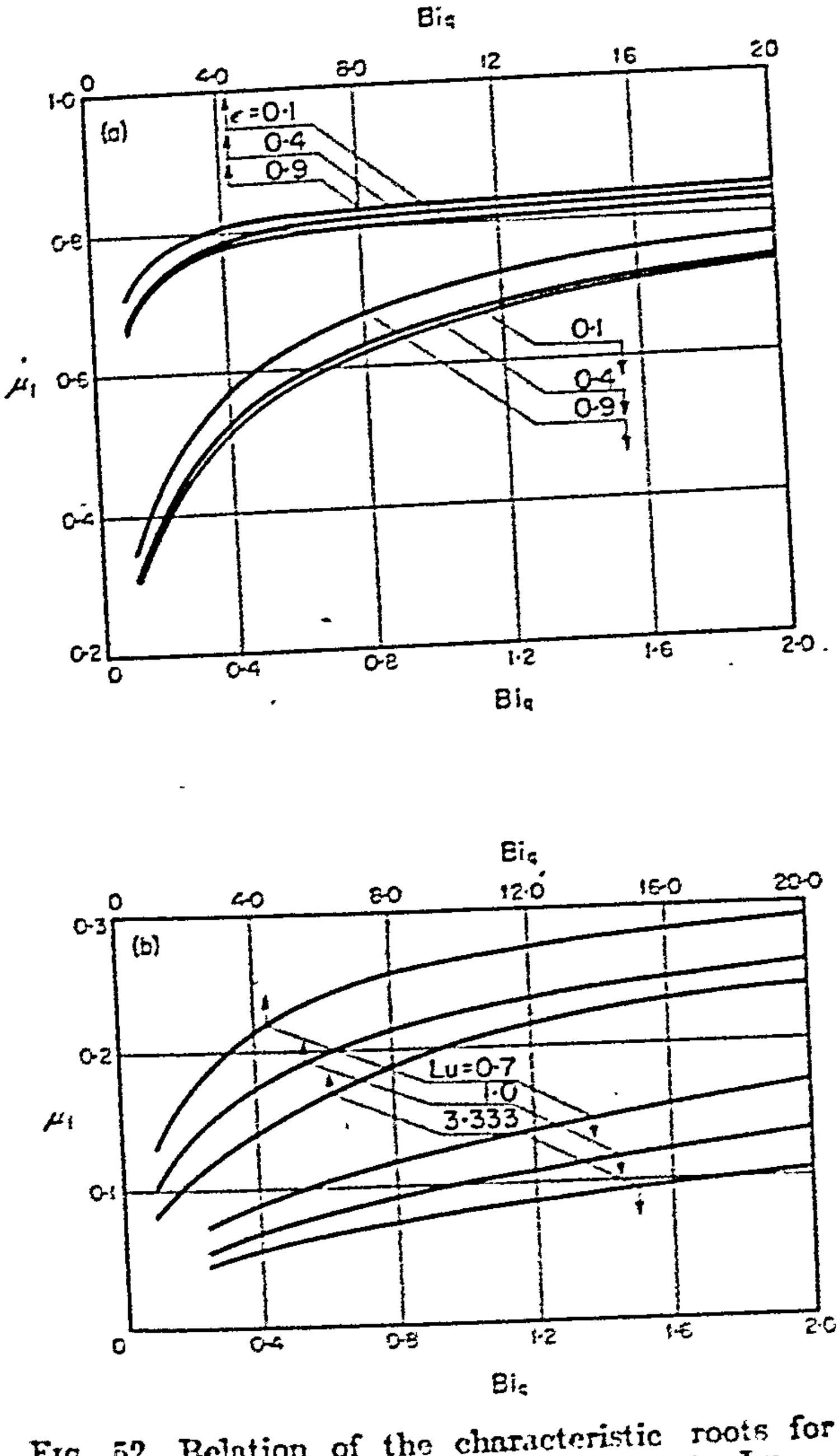


FIG. 52. Relation of the characteristic roots for an infinite plate: (*a*) — to ε; (*b*) — to Lu

nfinite series which enter into the solutions converge fairly rapidly. Beginning with a certain value of the Fourier criterion, it is necessary to use only the first two or three terms of the whole series. On the average, beginning from the value of Fo = 0·7, the error in the calculation, introduced by using

only the first two or three terms of the series, does not exceed 1 or 2 per cent. On the basis of what has been said above, for practical calculations the solutions (1.41)–(1.43) can be simplified and presented in the following form:

$$\overline{T} \approx 1 - \sum_{n=1}^{2} D_{nT} \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.44}$$

$$\overline{\Theta} \approx 1 + \sum_{n=1}^{2} D_{n\Theta} \exp(-\mu_n^2 \mathrm{Fo}) \tag{1.45}$$

$$\overline{P} \approx \sum_{n=1}^{2} D_{np} \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.46}$$

where

$$D_{nT} = \sum_{j=1}^{3} D_{nj}, \quad D_{n\Theta} = \frac{1}{\varepsilon \mathrm{Ko}} \sum_{j=1}^{3} D_{nj}(1 - \nu_j^2),$$

$$D_{np} = -\frac{1}{\varepsilon \mathrm{Bu}} \sum_{j=1}^{3} D_{nj} \sigma_j.$$

The values of the coefficients C_{nj}, D_{nj} and also D_{nT}, $D_{n\Theta}$, D_{np} for different combinations of the criteria of heat and mass exchange are shown in Tables 42–46.

(b) *Sphere.* The system of differential equations for the high intensity heat and mass transfer (XVIII) for a sphere can be rewritten in the following form:

$$\frac{\partial [XT]}{\partial \mathrm{Fo}} = \frac{\partial^2 [XT]}{\partial X^2} - \varepsilon \mathrm{Ko} \frac{\partial [X\Theta]}{\partial \mathrm{Fo}}, \tag{1.47}$$

$$\frac{\partial [X\Theta]}{\partial \mathrm{Fo}} = \mathrm{Lu} \frac{\partial^2 [X\Theta]}{\partial X^2} - \mathrm{Lu}\,\mathrm{Pn} \frac{\partial^2 [XT]}{\partial X^2} - \mathrm{Lu}_p \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\partial^2 [XP]}{\partial X^2}, \tag{1.48}$$

$$\frac{\partial [XP]}{\partial \mathrm{Fo}} = \mathrm{Lu}_p \frac{\partial^2 [XP]}{\partial X^2} + \varepsilon \frac{\mathrm{Ko}}{\mathrm{Bu}} \frac{\partial [X\Theta]}{\partial \mathrm{Fo}}. \tag{1.49}$$

Let us write

$$XT = T^*, \; X\Theta = \Theta^*, \; XP = P^*. \tag{1.50}$$

TABLE 42. VALUES OF THE CONSTANT COEFFICIENTS FOR AN INFINITE PLATE.
(Lu = 0·3, ε = 0·7, Bi$_m$ = 20, Ko = 9, Pn = 0·25, Lu$_p$ = 500, Bu = 10^{-3})

	Bi$_q$ = 5	Bi$_q$ = 10	Bi$_q$ = 15	Bi$_q$ = 20	Bi$_q$ = 30		
	μ_1	μ_1	μ_1	μ_1	μ_1	μ_2	μ_3
C_1	3·053	3·052	3·052	3·054	3·051	—1·586	0·6413
C_2	—0·6316×10^{-2}	—0·1760×10^{-2}	—0·1822×10^{-3}	0·6397×10^{-3}	0·1437×10^{-2}	0·4536×10^{-2}	0·1186×10^{-2}
C_3	0·7861	0·2965	0·1252	0·3672×10^{-1}	—0·05070	—0·9590	—0·4161
D_1	2·118	2·073	2·057	2·051	2·041	+0·3424	—0·1370
D_2	—0·6311×10^{-2}	—0·1759×10^{-2}	—0·1821×10^{-3}	0·6393×10^{-3}	0·1436×10^{-2}	0·4507×10^{-2}	0·1177×10^{-2}
D_3	0·7617	0·2868	0·1210	0·3549×10^{-1}	—0·04899	—0·6992	—0·2894
$D_{n\Gamma}$	2·873	2·358	2·178	2·087	1·993	—0·3523	—0·4252
$D_{n\theta}$	—0·6792	—0·7149	—0·7271	—0·7344	—0·7400	—0·2000	0·01746
D_{np}	2·420	2·657	2·746	2·790	2·834	1·910	0·6403

Bi_m

TABLE 43. VALUES OF THE CONSTANT COEFFICIENTS FOR AN INFINITE PLATE
($Lu = 0{\cdot}3$, $\varepsilon = 0{\cdot}7$, $Bi_q = 20$, $Ko = 9$, $Pn = 0{\cdot}25$, $Lu_p = 500$, $Bu = 10^{-3}$)

	$Bi_m = 5$			$Bi_m = 8$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
C_1	3·022	—0·6713	—0·1646	3·032	—0·8157	$-0{\cdot}3260\times10^{-1}$
C_2	$0{\cdot}8495\times10^{-2}$	$0{\cdot}9549\times10^{-2}$	$-0{\cdot}2836\times10^{-3}$	$0{\cdot}4320\times10^{-2}$	$0{\cdot}8902\times10^{-2}$	$-0{\cdot}1388\times10^{-4}$
C_3	—0·5517	—1·221	0·1844	—0·2480	—1·337	$0{\cdot}4045\times10^{-4}$
D_1	2·218	$0{\cdot}4870\times10^{-1}$	$0{\cdot}3565\times10^{-1}$	2·120	0·1233	$-0{\cdot}7061\times10^{-2}$
D_2	$0{\cdot}8480\times10^{-2}$	$0{\cdot}9512\times10^{-2}$	$-0{\cdot}2816\times10^{-3}$	$0{\cdot}4317\times10^{-2}$	$0{\cdot}8660\times10^{-2}$	$-0{\cdot}1378\times10^{-4}$
D_3	—0·5370	—1·016	0·1305	—0·2404	—1·067	$0{\cdot}3500\times10^{-1}$
D_{nT}	1·089	—0·958	0·1650	1·884	—0·935	$0{\cdot}2702\times10^{-1}$
$D_{n\Theta}$	—0·8565	—0·1280	$0{\cdot}1570\times10^{-2}$	—0·7892	—0·1611	$0{\cdot}6416\times10^{-3}$
D_{np}	2·437	0·8314	—0·4340	2·646	1·459	0·1083

TABLE 43. *Contd.*

	$Bi_m = 15$			$Bi_m = 20$
	μ_1	μ_2	μ_3	μ_1
C_1	3·040	—1·977	0·9574	3·054
C_2	$0·1445 \times 10^{-2}$	$0·6335 \times 10^{-2}$	$0·5966 \times 10^{-3}$	$0·6307 \times 10^{-3}$
C_3	$-0·2636 \times 10^{-1}$	—1·114	—0·5776	$0·3672 \times 10^{-1}$
D_1	2·066	0·4211	—0·2079	2·051
D_2	$0·1444 \times 10^{-2}$	$0·6296 \times 10^{-2}$	$0·5925 \times 10^{-3}$	$0·6303 \times 10^{-3}$
D_3	$-0·2549 \times 10^{-1}$	—0·8228	—0·4114	$0·3549 \times 10^{-1}$
D_{nm}	2·042	—0·3054	—0·6187	2·087
D_{n0}	—0·7463	—0·2417	$0·2037 \times 10^{-1}$	—0·7344
D_{np}	2·761	1·860	1·341	2·790

Ko

TABLE 45. VALUES OF THE CONSTANT COEFFICIENTS FOR AN INFINITE PLATE
($\mathrm{Lu} = 0 \cdot 3$, $\varepsilon = 0 \cdot 7$, $\mathrm{Bi}_q = 20$, $\mathrm{Bi}_m = 20$, $\mathrm{Pn} = 0 \cdot 25$, $\mathrm{Lu}_p = 500$, $\mathrm{Bu} = 1 \times 10^{-3}$)

	Ko = 3		Ko = 6		
	μ_1	μ_2	μ_1	μ_2	μ_3
C_1	2·199	—0·1131	2·721	—0·3165	—0·5500
C_2	$0 \cdot 8457 \times 10^{-2}$	$0 \cdot 1013 \times 10^{-3}$	$0 \cdot 7109 \times 10^{-3}$	$0 \cdot 2889 \times 10^{-2}$	$0 \cdot 4604 \times 10^{-3}$
C_3	—0·067772	—0·8361	$0 \cdot 4196 \times 10^{-2}$	1·383	$0 \cdot 3319 \times 10^{-2}$
D_1	1·407	$-0 \cdot 1195 \times 10^{-1}$	1·832	$0 \cdot 4725 \times 10^{-1}$	0·1192
D_2	$0 \cdot 8447 \times 10^{-3}$	$0 \cdot 1009 \times 10^{-2}$	$0 \cdot 7103 \times 10^{-3}$	$0 \cdot 2872 \times 10^{-2}$	$0 \cdot 4567 \times 10^{-3}$
D_3	—0·0608	—0·5911	$0 \cdot 3988 \times 10^{-2}$	—0·9797	$0 \cdot 2006 \times 10^{-2}$
D_{nT}	1·436	—0·6020	1·835	—0·9296	0·1217
$D_{n\Theta}$	—0·7505	—0·1376	—0·7305	—0·1651	$-0 \cdot 1733 \times 10^{-1}$
D_{np}	1·313	0·8053	1·979	2·226	$-0 \cdot 5047 \times 10^{-1}$

Table 45. *Contd.*

	Ko = 0	Ko = 12		
	μ_1	μ_1	μ_2	μ_3
C_1	3·051	3·201	—1·557	0·5701
C_2	$0{\cdot}6397\times10^{-3}$	$-0{\cdot}2814\times10^{-3}$	$-0{\cdot}1577\times10^{-3}$	$0{\cdot}8058\times10^{-3}$
C_3	$0{\cdot}3072\times10^{-4}$	$0{\cdot}9119\times10^{-4}$	0·4718	—2·460
D_1	2·051	2·102	0·3379	$-0{\cdot}8060\times10^{-4}$
D_2	$0{\cdot}6393\times10^{-3}$	$-0{\cdot}2812\times10^{-3}$	$-0{\cdot}1568\times10^{-3}$	$0{\cdot}7003\times10^{-3}$
D_3	$0{\cdot}3549\times10^{-4}$	$0{\cdot}8803\times10^{-4}$	0·3708	—1·750
$D_{n\Gamma}$	2·087	2·281	0·7071	—1·829
D_{n0}	—0·7344	—0·7330	$-0{\cdot}8157\times10^{-4}$	—0·1246
D_{np}	2·700	3·037	—0·0740	0·859

Lu_p

TABLE 46. Lu_p VALUES OF THE CONSTANT COEFFICIENTS FOR AN INFINITE PLATE
($Lu = 0{\cdot}3$, $\varepsilon = 0{\cdot}7$, $Bi_q = 20$, $Bi_m = 20$, $Bu^{-3} = 10$, $Ko = 9$, $Pn = 0{\cdot}25$)

	$Lu_p = 300$	$Lu_p = 500$	$Lu_p = 700$	$Lu_p = 1000$
	μ_1	μ_1	μ_1	μ_1
C_1	3·031	3·054	3·073	3·040
C_2	$0{\cdot}1588\times10^{-2}$	$0{\cdot}6397\times10^{-3}$	$0{\cdot}2718\times10^{-3}$	$0{\cdot}1887\times10^{-3}$
C_3	$0{\cdot}3014\times10^{-1}$	$0{\cdot}3672\times10^{-1}$	$0{\cdot}4202\times10^{-1}$	$0{\cdot}5231\times10^{-1}$
D_1	2·038	2·051	2·063	2·033
D_2	$0{\cdot}1587\times10^{-2}$	$0{\cdot}6393\times10^{-3}$	$0{\cdot}2717\times10^{-3}$	$0{\cdot}1886\times10^{-3}$
D_3	$0{\cdot}2913\times10^{-1}$	$0{\cdot}3549\times10^{-1}$	$0{\cdot}4061\times10^{-3}$	$0{\cdot}5053\times10^{-1}$
D_{nT}	2·069	2·087	2·104	2·084
$D_{n\theta}$	—0·7298	—0·7344	—0·7381	—0·7207
D_{np}	4·931	2·790	1·609	1·371

In this form the system of equations is identical with that for an infinite plate. Consequently, in representation, its solution is

$$T_L^* = A_1 \exp[\sqrt{(s)}\, r_1 X] + A_2 \exp[\sqrt{(s)}\, r_2 X] +$$
$$+ A_3 \exp[\sqrt{(s)}\, r_3 X] + A_4 \exp[-\sqrt{(s)}\, r_1 X] +$$
$$+ A_5 \exp[-\sqrt{(s)}\, r_2 X] + A_6 \exp[-\sqrt{(s)}\, r_2 X], \quad (1.51)$$

$$\Theta_L^* = -\frac{1}{\varepsilon \mathrm{Ko}}\{A_1(1 - r_1^2) \exp[\sqrt{(s)}\, r_1 X] +$$
$$+ A_2(1 - r_2^2) \exp[\sqrt{(s)}\, r_2 X] + A_3(1 - r_3^2) \exp[\sqrt{(s)}\, r_3 X] +$$
$$+ A_4(1 - r_1^2) \exp[-\sqrt{(s)}\, r_1 X] + A_5(1 - r_2^2) \exp[-\sqrt{(s)}\, r_2 X] +$$
$$+ A_6(1 - r_3^2) \exp[-\sqrt{(s)}\, r_3 X]\}, \quad (1.52)$$

$$P_L^* = \frac{1}{\varepsilon \mathrm{Bu}}\{A_1 \sigma_1 \exp[\sqrt{(s)}\, r_1 X] + A_2 \sigma_2 \exp[\sqrt{(s)}\, r_2 X] +$$
$$+ A_3 \sigma_3 \exp[\sqrt{(s)}\, r_3 X] + A_4 \sigma_1 \exp[-\sqrt{(s)}\, r_1 X] +$$
$$+ A_5 \sigma_2 \exp[-\sqrt{(s)}\, r_2 X] + A_6 \sigma_3 \exp[-\sqrt{(s)}\, r_3 X]\}, \quad (1.53)$$

where σ_j and r_j are respectively determined by (1.13) and (1.14).

The conditions of symmetry for the sphere (1.4) on the basis of the known transformations can be written thus:

$$T'_L(0, s) = \Theta'_L(0, s) = P'_L(0, s) = 0. \quad (1.54)$$

Adjusting (1.51) to (1.53) to the conditions (1.54) we obtain

$$A_1 = -A_4, \quad A_2 = -A_5, \quad A_3 = -A_6.$$

Consequently the solution of the problem in transformation has the following general form:

$$T_L^*(X, s) = \sum_{j=1}^{3} C_j^* \sinh \sqrt{(s)}\, r_j X, \quad (1.55)$$

$$\Theta_L^*(X, s) = -\frac{1}{\varepsilon \mathrm{Ko}} \sum_{j=1}^{3} C_j^*(1 - r_j^2) \sinh \sqrt{(s)}\, r_j X, \quad (1.56)$$

$$P_L^*(X, s) = \frac{1}{\varepsilon \mathrm{Bu}} \sum_{j=1}^{3} C_j^* \sigma_j \sinh \sqrt{(s)}\, r_j X. \quad (1.57)$$

Substituting (1.55)–(1.57) in the boundary conditions (1.18)–(1.20), and remembering (1.50), we obtain the following algebraic system of equations:

$$\left.\begin{aligned} &\sum_{j=1}^{3} C_j^* P_j^* + \frac{\varepsilon \,\mathrm{Ko}}{s} = 0 \\ &\sum_{j=1}^{3} C_j^* Q_j^* + \frac{\varepsilon \,\mathrm{Ko}\, K_1 - 1}{s} = 0 \\ &\sum_{j=1}^{3} C_j^* \chi_j^* = 0 \end{aligned}\right\}, \tag{1.58}$$

where

$$Q_j^* = \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2)\, K_1\right] \sinh \sqrt{(s)}\, \nu_j + \\ + \frac{1}{\mathrm{Bi}_q} \sqrt{(s)}\, \nu_j \cosh \sqrt{(s)}\, \nu_j ,$$

$$P_j^* = \frac{(1 - \nu_j^2) + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m} \left[\sqrt{(s)}\nu_j \cosh \sqrt{(s)}\nu_j \sinh \sqrt{(s)}\nu_j\right] + \\ + (1 - \nu_j^2) \sinh \sqrt{(s)}\, \nu_j ,$$

$$\chi_j^* = \sigma_j \sinh \sqrt{(s)}\, \nu_j .$$

Their solution may be written thus:

$$C_j^* = \left[(\varepsilon \,\mathrm{Ko}\, K_1 - 1)\left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^*\right) - \varepsilon \,\mathrm{Ko}\left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q_3^*\right)\right] \times \\ \times \left\{s\left[\left(P_1^* - \frac{\chi_1^*}{\chi_3^*}\right)\left(Q_2^* - \frac{\chi_2^*}{\chi_2^*} Q_3^*\right) - \right.\right. \\ \left.\left. - \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^*\right)\left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^*\right)\right]\right\}^{-1},$$

$$C_2^* = -\left[(\varepsilon \,\mathrm{Ko}\, K_1 - 1)\left(P_1^* - \frac{\chi_1^*}{\chi_3^*} P_3^*\right) - \varepsilon \,\mathrm{Ko}\left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^*\right)\right] \times \\ \times \left\{s\left[\left(P_3^* - \frac{\chi_1^*}{\chi_3^*} P_3^*\right)\left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q^*\right) - \right.\right. \\ \left.\left. - \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^*\right)\left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^*\right)\right]\right\}^{-1}.$$

$$O_3^* = \left[(\varepsilon \mathrm{Ko} K_1 - 1)\left(P_1^* \frac{\chi_2^*}{\chi_3^*} - P_2^* \frac{\chi_1^*}{\chi_3^*}\right) - \varepsilon \mathrm{Ko}\left(Q_1^* \frac{\chi_2^*}{\chi_3^*} - Q_2^* \frac{\chi_1^*}{\chi_3^*}\right)\right] \times$$

$$\times \left\{ s \left[\left(P_1^* - \frac{\chi_1^*}{\chi_3^*} P_3^*\right)\left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q_3^*\right) - \right.\right.$$

$$\left.\left. - \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^*\right)\left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^*\right)\right]\right\}^{-1}.$$

Thus the problem under consideration has been solved in transform.

The inversions of (1.55)–(1.57) can be obtained by the expansion theorem, transforming the numerator and denominator of the corresponding expressions into generalized polynomials. Since the method of transformation of generalized polynomials for the sphere has been several times considered in the preceding chapters, we give here only the final solutions of the problem:

$$T^* = X + \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \sin \nu_j \mu_n X \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.59)$$

$$\Theta^* = X - \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj}(1 - \nu_j^2) \sin \nu_j \mu_n \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.60)$$

$$P^* = \frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \sigma_j \sin \nu_j \mu_n X \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.61)$$

or, taking into account (1.50)

$$T = 1 + \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \frac{\sin \nu_j \mu_n X}{X} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.59')$$

$$\Theta = 1 - \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj}(1 - \nu_j^2) \frac{\sin \nu_j \mu_n X}{X} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \qquad (1.60')$$

$$P = \frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \sigma_j \frac{\sin \nu_j \mu_n X}{X} \exp\left(-\mu_n^2 \mathrm{Fo}\right). \qquad (1.61')$$

Here C_{nj} and ν_n are determined in form respectively by the expressions (1.30)–(1.33), the characteristic roots μ_n by

equation (1.40) and

$$A_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n - \frac{1}{\mathrm{Bi}_q} \nu_j \mu_n \sin \nu_j \mu_n ,$$

$$B_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n -$$

$$- \frac{(1 - \nu_j^2) + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} \nu_j \mu_n \sin \nu_j \mu_n ,$$

$$Q_{nj} = \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n ,$$

$$P_{nj} = - \frac{(1 - \nu_j^2) + \varepsilon \,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} (\nu_j \mu_n \cos \nu_j \mu_n - \sin \nu_j \mu_n) +$$

$$+ (1 - \nu_j^2) \sin \nu_j \mu_n , \qquad \chi_{nj} = \sigma_j \sin \nu_j \mu_n .$$

The values of the characteristic roots μ_n and the coefficients C_{nl} for different combinations of the criteria of heat and mass exchange are shown in Tables 47–53. The mean values of the non-dimensional transfer potentials for a sphere can be obtained from the formula

$$\bar{\varphi}(\mathrm{Fo}) = 3 \int_0^1 X^2 \varphi(X, \mathrm{Fo})\, dX .$$

TABLE 47. RELATION OF THE CHARACTERISTIC ROOTS OF EQUATION (1.40) TO Lu AND Bi_q FOR A SPHERE ($\mathrm{Lu}_p = 500$, $\varepsilon = 0.5$, Kv = 9, Pn = 0.25, $\mathrm{Bi}_m = 20$)

Lu	Bi_q	μ_1	μ_2	μ_3
0·25	5	1·5371	3·1790	3·6256
	10	1·5707	3·1877	3·8489
	15	1·5817	3·1896	3·9186
	20	1·6823	3·3826	4·2011
	30	1·5926	3·9818	4·7803
0·50	5	1·6984	3·5610	6·3729
	10	1·7661	3·6060	7·0806
	15	1·7897	3·6224	7·1711
	20	1·8017	3·6309	7·2194
	30	1·8138	3·6396	4·9790

Lu	Bi_T	μ_1	μ_2	μ_3
0·75	5	1·7824	3·6794	6·9708
	10	1·8626	3·7727	7·3356
	15	1·8908	3·8091	7·4943
	20	1·9051	3·8287	7·5758
	30	1·9196	3·8490	7·6767
1·00	5	1·8142	3·7460	6·0707
	10	1·9028	3·8512	6·0376
	15	1·9342	3·8942	6·0184
	20	1·9503	3·9176	6·0061
	30	1·9666	3·9423	5·9915
2·00	5	1·8611	3·8460	5·9829
	10	1·9630	3·9693	6·0571
	15	1·9997	4·0223	6·0965
	20	2·0186	4·0517	6·1210
	30	2·0379	4·0833	6·1499

TABLE 48. RELATION OF THE CHARACTERISTIC ROOTS OF EQUATION (1.40) TO Bi_m, ε, Pn, Ko, Lu_p FOR A SPHERE

Criterion		μ_1	μ_2	μ_3	Criterion		μ_1	μ_2	μ_3
Bi_m	5	1·7034	3·3876	4·3156	Lu_p	100	1·8022	3·6329	7·2173
	10	1·7801	3·5820	4·7356		250	1·8018	3·6312	7·2190
	15	1·8050	3·6328	4·9722		500	1·8017	3·6309	7·2194
	20	1·8017	3·6309	7·2194		750	1·8016	3·6308	7·2196
	30	1·8294	3·6746	7·3190		1000	1·8016	3·6308	7·2197
ε	0·25	1·9920	5·7124	9·5451	Ko	3	2·2100	4·3909	6·6531
	0·50	1·8017	3·6309	7·2194		6	1·9747	3·0927	4·4904
	0·75	1·7518	3·5115	5·3046		9	1·8017	3·6309	7·2194
	1·00					12	1·6984	3·4108	6·7488
Pn	0·05	2·3538	3·7291	4·7049					
	0·25	1·8017	3·6309	7·2194					
	0·45	1·5698	3·1494	4·7738					
	0·65	1·4089	2·8240	4·2562					

TABLE 49. Bi_q AND Lu VALUES OF THE CONSTANT COEFFICIENTS FOR A SPHERE
(Lu = 500, $\varepsilon = 0{\cdot}5$, Ko = 9, Pn = 0·25, $Bi_m = 20$)

Lu	Bi_q	Cn_1			Cn_2			Cn_3		
		$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0·25	5	−1·1209	0·7241	−0·2129	$0{\cdot}6621\times10^{-2}$	$0{\cdot}7936\times10^{-2}$	$-0{\cdot}5789\times10^{-2}$	−0·8998	−2·3373	2·6150
	10	−1·1064	0·6134	−0·0899	$0{\cdot}1816\times10^{-2}$	$0{\cdot}1858\times10^{-2}$	$-0{\cdot}1389\times10^{-2}$	−0·3683	−0·7048	0·9099
	15	−1·1006	0·5881	−0·0661	$0{\cdot}2901\times10^{-3}$	$0{\cdot}5415\times10^{-3}$	$-0{\cdot}8254\times10^{-3}$	−0·1971	−0·3471	0·6231
	20	−1·0251	0·5362	−0·0533	$-0{\cdot}4525\times10^{-3}$	$0{\cdot}8451\times10^{-4}$	$-0{\cdot}5100\times10^{-3}$	−0·1134	−0·1989	0·3761
	30	−1·0943	−0·0512	−0·3164	$-0{\cdot}1192\times10^{-2}$	$-0{\cdot}5153\times10^{-3}$	$-0{\cdot}3605\times10^{-3}$	−0·0295	0·4491	0·0381
0·50	5	−1·5175	0·9075	0·1849	$0{\cdot}2660\times10^{-1}$	$0{\cdot}1507\times10^{-1}$	$0{\cdot}1557\times10^{-2}$	−1·1278	−1·2437	0·2904
	10	−1·4729	0·8413	0·2757	$0{\cdot}1038\times10^{-1}$	$0{\cdot}6568\times10^{-2}$	$0{\cdot}1305\times10^{-2}$	−0·4915	−0·6006	0·2538
	15	−1·4560	0·8155	0·2988	$0{\cdot}5093\times10^{-2}$	$0{\cdot}3444\times10^{-2}$	$0{\cdot}7651\times10^{-3}$	−0·2799	−0·3554	0·1714
	20	−1·4471	0·8018	0·3072	$0{\cdot}2483\times10^{-2}$	$0{\cdot}1824\times10^{-2}$	$0{\cdot}3882\times10^{-3}$	−0·1746	−0·2263	0·1146
	30	−1·4379	0·7875	−0·2588	$-0{\cdot}9954\times10^{-4}$	$0{\cdot}1638\times10^{-3}$	$-0{\cdot}2328\times10^{-2}$	−0·0699	−0·0926	0·7600
0·75	5	−1·6285	0·8723	−0·0253	$0{\cdot}2733\times10^{-1}$	$0{\cdot}1254\times10^{-1}$	$-0{\cdot}1535\times10^{-3}$	−1·1036	−0·8452	−0·0525
	10	−1·5764	0·8455	0·2306	$0{\cdot}1198\times10^{-2}$	$0{\cdot}6301\times10^{-2}$	$0{\cdot}1114\times10^{-2}$	−0·5051	−0·4541	0·2852
	15	−1·5560	0·8297	0·2880	$0{\cdot}6995\times10^{-2}$	$0{\cdot}3875\times10^{-2}$	$0{\cdot}9268\times10^{-3}$	−0·3077	−0·2931	0·2249
	20	−1·5451	0·8241	0·3079	$0{\cdot}4549\times10^{-2}$	$0{\cdot}2591\times10^{-2}$	$0{\cdot}6701\times10^{-3}$	−0·2099	−0·2059	0.1649
	30	−1·5338	0·8154	0·3222	$0{\cdot}2137\times10^{-2}$	$0{\cdot}1268\times10^{-2}$	$0{\cdot}4045\times10^{-3}$	−0·1131	−0·1143	0·1016
1·00	5	−1·6886	0·8960	−1·1203	$0{\cdot}3735\times10^{-1}$	$0{\cdot}1576\times10^{-1}$	$0{\cdot}1140\times10^{-1}$	−1·1665	−0·7385	−2·0658
	10	−1·6359	0·8687	−1·0331	$0{\cdot}1688\times10^{-1}$	$0{\cdot}8321\times10^{-2}$	$0{\cdot}8247\times10^{-2}$	−0·5422	−0·4093	−1·4526
	15	−1·6114	0·8540	−0·9627	$0{\cdot}1024\times10^{-1}$	$0{\cdot}5329\times10^{-2}$	$0{\cdot}6059\times10^{-2}$	−0·3365	−0·2706	−1·0562
	20	−1·5984	0·8454	−0·9127	$0{\cdot}6973\times10^{-2}$	$0{\cdot}3735\times10^{-2}$	$0{\cdot}4581\times10^{-2}$	−0·2348	−0·1950	−0·7980
	30	−1·5847	0·8357	−0·8501	$-0{\cdot}3757\times10^{-2}$	$0{\cdot}2076\times10^{-2}$	$0{\cdot}2781\times10^{-2}$	−0·1341	−0·1151	−0·4926
2·00	5	−1·8002	0·9316	−0·7244	$0{\cdot}7780\times10^{-1}$	$0{\cdot}2971\times10^{-1}$	$0{\cdot}1558\times10^{-1}$	−1·4143	−0·6696	−0·5605
	10	−1·7180	0·9002	−0·6829	$0{\cdot}3654\times10^{-1}$	$0{\cdot}1664\times10^{-1}$	$0{\cdot}1005\times10^{-1}$	−0·6715	−0·3838	−0·3711
	15	−1·6859	0·8821	−0·6606	$0{\cdot}2314\times10^{-1}$	$0{\cdot}1119\times10^{-1}$	$0{\cdot}7272\times10^{-2}$	−0·4282	−0·2612	−0·2727
	20	−1·6686	0·8709	−0·6468	$0{\cdot}1657\times10^{-1}$	$0{\cdot}8247\times10^{-2}$	$0{\cdot}5603\times10^{-2}$	−0·3084	−0·1943	−0·2126
	30	−1·6505	0·8579	−0·6305	$0{\cdot}1011\times10^{-1}$	$0{\cdot}5176\times10^{-2}$	$0{\cdot}3699\times10^{-2}$	−0·1903	−0·1238	−0·1428

Bi_m

TABLE 50. ($Lu = 0.5$, $\epsilon = 0.5$, $Pn = 0.25$, $Ko = 0$, $Lu_p = 500$, $Bi_q = 20$)

Bi_m	Co_1			Co_2			Co_3		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
5	-1.5162	0·8308	-0.2901	-0.0860×10^{-2}	-0.4351×10^{-3}	-0.3486×10^{-2}	0·1478	0·2090	0·5501
10	-1.4051	0·8352	-0.3158	-0.6237×10^{-2}	-0.3431×10^{-1}	-0.3069×10^{-2}	-0.7750×10^{-1}	-0.1120	0.8521
15	-1.4126	0·7959	-0.2540	0.1208×10^{-2}	0.9036×10^{-3}	-0.1614×10^{-2}	-0.1445	-0.1950	0·7582
20	-1.4171	0·8018	0·3072	0.2483×10^{-2}	0.1824×10^{-2}	0.3882×10^{-3}	-0.1746	-0.2263	0·1140
30	-1.4170	0·7466	0·3320	0.2801×10^{-2}	0.1701×10^{-2}	0.608×210^{-3}	-0.2063	-0.2526	0·1643

Pn

TABLE 51. VALUES OF THE CONSTANT COEFFICIENTS FOR A SPHERE

($Lu = 0.5$, $\epsilon = 0.5$, $Ko = 0$, $Lu_p = 500$, $Bi_q = 20$, $Bi_m = 20$)

Pn	Co_1			Co_2			Co_3		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0·05	-3.4902	0·3098	1·4443	-0.1322×10^{-2}	-0.2581×10^{-2}	-0.6180×10^{-3}	-0.1310	2·0614	0·1281
0·25	-1.4171	0·8018	0·3072	0.2183×10^{-2}	0.1824×10^{-2}	0.3882×10^{-3}	-0.1740	-0.2263	0·1140
0·45	-0.9872	0·5007	-0.3771	0.4017×10^{-2}	0.2403×10^{-2}	0.2013×10^{-2}	-0.1053	-0.1440	-0.2717
0·65	-0.7680	0·3841	-0.2000	0.6786×10^{-2}	0.3374×10^{-2}	0.2280×10^{-2}	-0.2005	-0.1205	0·1344

Ko

TABLE 52. VALUES OF THE CONSTANT COEFFICIENTS FOR A SPHERE
(Lu = 0.5, $\varepsilon = 0.5$, Pn = 0.25, $Lu_p = 500$, $Bi_q = 20$, $Bi_m = 20$)

Ko	Cn_1			Cn_2			Cn_3		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
3	−1·0298	0·4465	−0·3279	$-0{\cdot}2423\times10^{-3}$	$-0{\cdot}2889\times10^{-3}$	$-0{\cdot}5912\times10^{-4}$	$-0{\cdot}5753\times10^{-1}$	0·1509	$0{\cdot}5498\times10^{-1}$
6	−1·2706	0·8023	−0·2000	$0{\cdot}5766\times10^{-3}$	$0{\cdot}9065\times10^{-3}$	$-0{\cdot}1519\times10^{-2}$	−0·1158	−0·3824	0·9403
9	−1·4471	0·8018	0·3072	$0{\cdot}2483\times10^{-2}$	$0{\cdot}1824\times10^{-2}$	$0{\cdot}3882\times10^{-3}$	−0·1746	−0·2263	0·1146
12	−1·5485	0·8115	0·3107	$0{\cdot}4125\times10^{-2}$	$0{\cdot}2354\times10^{-2}$	$0{\cdot}5286\times10^{-3}$	−0·2390	−0·2232	0·2010

Lu_p

TABLE 53. VALUES OF THE CONSTANT COEFFICIENTS FOR A SPHERE
(Lu = 0.5, $\varepsilon = 0.5$, Pn = 0.25, Ko = 0, $Bi_q = 20$, $Bi_m = 20$)

Lu_p	Cn_1			Cn_2			Cn_3		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
100	−1·4493	0·8028	0·2987	$0{\cdot}6509\times10^{-2}$	$0{\cdot}4829\times10^{-2}$	$0{\cdot}1101\times10^{-2}$	−0·1768	−0·2252	0·1122
250	−1·4477	0·8020	0·3053	$0{\cdot}3649\times10^{-2}$	$0{\cdot}2687\times10^{-2}$	$0{\cdot}5820\times10^{-3}$	−0·1751	−0·2260	0·1140
500	−1·4471	0·8018	0·3072	$0{\cdot}2483\times10^{-2}$	$0{\cdot}1824\times10^{-2}$	$0{\cdot}3882\times10^{-3}$	−0·1746	−0·2263	0·1146
750	−1·4469	0·8017	0·3078	$0{\cdot}2002\times10^{-2}$	$0{\cdot}1469\times10^{-2}$	$0{\cdot}3109\times10^{-3}$	−0·1745	−0·2264	0·1148
1000	−1·4468	0·8016	0·3081	$0{\cdot}1722\times10^{-2}$	$0{\cdot}1264\times10^{-2}$	$0{\cdot}2668\times10^{-3}$	−0·1744	−0·2265	0·1148

$$P(X,\mathrm{Fo}) = -\frac{1}{\varepsilon\,\mathrm{Bu}}\sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj}\,\sigma_j\,I_0(\nu_j\,\mu_n\,X)\exp\left(-\mu_n^2\,\mathrm{Fo}\right).$$

Here C_{nj} and ψ_n are determined in the form respectively by the expressions (1.30)–(1.33), σ_j and ν_j by the expressions (1.13) and (1.14), the characteristic roots μ_n by equation (1.40), and

$$Q_{nj} = [1 + K_1(1-\nu_j^2)]\,I_0(\nu_j\,\mu_n) - \frac{1}{\mathrm{Bi}_q}\nu_j\mu_n I_1(\nu_j\mu_n),$$

$$P_{nj} = (1-\nu_j^2)\,I_0(\nu_j\,\mu_n) - \frac{(1-\nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}\nu_j\mu_n I_1(\nu_j\,\mu_n),$$

$$A_{nj} = [1 + K_1(1-\nu_j^2)]\,I_1(\nu_j\,\mu_n) + \frac{1}{\mathrm{Bi}_q}\nu_j\,\mu_n\,I_0(\nu_j\,\mu_n),$$

$$B_{nj} = (1-\nu_j^2)\,I_1(\nu_j\,\mu_n) + \frac{(1-\nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}\nu_j\mu_n I_0(\nu_j\mu_n),$$

$$b_{nj} = \nu_j\frac{I_1(\nu_j\,\mu_n)}{I_0(\nu_j\,\mu_n)} - \nu_3\frac{I_1(\nu_3\,\mu_n)}{I_0(\nu_3\,\mu_n)}.$$

The mean values of non-dimensional transfer potentials for a cylinder can be obtained from the formula

$$\overline{\varphi}(\mathrm{Fo}) = 2\int_0^1 X\,\varphi(X,\mathrm{Fo})\,dX.$$

Hence, we have

$$\overline{T} = 1 - \sum_{n=1}^{\infty}\sum_{j=1}^{3} D_{nj}\exp\left(-\mu_n^2\,\mathrm{Fo}\right),$$

$$\overline{\Theta} = 1 + \frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{n=1}^{\infty}\sum_{j=1}^{3}{}' D_{nj}(1-\nu_j^2)\exp\left(-\mu_n^2\,\mathrm{Fo}\right),$$

$$\overline{P} = -\frac{1}{\varepsilon\,\mathrm{Bu}}\sum_{n=1}^{\infty}\sum_{j=1}^{3} D_{nj}\,\sigma_j\exp\left(-\mu_n^2\,\mathrm{Fo}\right),$$

where

$$D_{nj} = 2C_{nj}\frac{I_1(\nu_j\,\mu_n)}{\nu_j\,\mu_n}.$$

§ 2. Non-stationary Fields of Transfer of Energy and Matter under Boundary Conditions of the Second Kind

In a number of high-intensity processes the specific flux of mass, evaporated from the material, $q_m(\tau)$, changes continuously with the time, and in this case the boundary conditions for uni-dimensional problems must be written as follows:

$$-a_c c_n \gamma_0 \frac{\partial t(R,\tau)}{\partial x} + \alpha[t_c - t(R,\tau)] - (1-\varepsilon)\varrho\, q_m(\tau) = 0\,,$$

$$a_m c_p \gamma_0 \frac{\partial \vartheta(R,\tau)}{\partial x} + a_m c_m \gamma_0 \delta \frac{\partial t(R,\tau)}{\partial x} +$$

$$+ a'_m c_m \gamma_0 \frac{\partial p(R,\tau)}{\partial x} + q_m(\tau) = 0,$$

$$p(R,\tau) = q(\tau)\,.$$

For the sake of generality we here consider separately the transfer of matter in the liquid form and transfer in the gaseous state. We shall do the same with the system of equations for high intensity heat and mass transfer, and also, in order to shorten the writing out of further transformations, we shall denote the transfer potentials by θ_i. so that $t = \theta_1$, $\theta = \theta_2$ and $p = \theta_3$. The system of differential equations and the limiting conditions. in view of what we have just said, will be written as follows:

$$\frac{\partial \theta_1}{\partial \tau} = a_1 \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\varepsilon\varrho\, c_2}{c_1} \frac{\partial \theta_2}{\partial \tau}\,, \tag{2.1}$$

$$\frac{\partial \theta_2}{\partial \tau} = a_2 \left(\delta \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_2}{\partial x^2}\right) + a'_3 \frac{\partial^2 \theta_3}{\partial x^2} + \varepsilon \frac{\partial \theta_2}{\partial \tau}\,, \tag{2.2}$$

$$\frac{\partial \theta_3}{\partial \tau} = a_3 \frac{\partial^2 \theta_3}{\partial x^2} + \varepsilon\gamma_0 \frac{c_2}{c_3} \frac{\partial \theta_2}{\partial \tau}\,, \tag{2.3}$$

$$\theta_i(x,0) = f_i(x)\,,$$
$$(0 < x < R)\; i = 1, 2, 3, \tag{2.4}$$

$$\frac{\partial \theta_i(0,\tau)}{\partial x} = 0\,. \tag{2.5}$$

$$- a_1 c_1 \gamma_0 \frac{\partial \theta_1(R,\tau)}{\partial x} \pm \alpha [\theta_{1c} - \theta_1(R,\tau)] -$$

$$- (1-\varepsilon) \varrho q_m(\tau) = 0 , \qquad (2.6)$$

$$a_2 c_3 \gamma_0 \frac{\partial \vartheta_2(R,\tau)}{\partial x} \pm a_2 c_2 \gamma_0 \delta \frac{\partial \theta_1(R.\tau)}{\partial x} \pm$$

$$\pm a_2' c_2 \gamma_0 \frac{\partial \vartheta_3(R,\tau)}{\partial x} \pm q_m(\tau) = 0 , \qquad (2.7)$$

$$\theta_3(R,\tau) = \varphi_3(\tau) . \qquad (2.8)$$

Here the subscripts 1, 2, 3 respectively denote q, m, p while a' is the transfer of matter in the liquid phase. The solution of this problem was given by Prudnikov. We give the general course of his solution.

First of all, let us consider an auxiliary problem. We will regard the functions $\theta_1(R,\tau) = \varphi_1(\tau)$, $\theta_2(R,\tau) = \varphi_2(\tau)$. as unknown, i. e.

$$\theta_k(R,\tau) = \varphi_k(\tau) \ (k = 1, 2) . \qquad (2.9)$$

These will be determined later. The solution of the auxiliary problem, consequently, reduces to the determination of the functions $\theta_i(x,\tau)$, which satisfy the system of equations (2.1)–(2.3), the initial conditions (2.4), the conditions of symmetry (2.5), and the boundary conditions (2.8), (2.9). We shall solve the problem by the method of simultaneous applicatio nof the integral transformations of Fourier and Laplace. With respect to the variable x. we use the finite cosine Fourier transformation:

$$\{\theta_i(p,\tau)\}_F = \int_0^R \theta_i(x,\tau) \cos\left(n \pm \frac{1}{2}\right) \frac{\pi}{R} x dx . \qquad (2.10)$$

The function $\theta_i(x,\tau)$ is determined from the known transform $\{\theta_i(p,\tau)\}_F$ by means of the series

$$\theta_i(x,\tau) = \frac{2}{R} \sum_{n=0}^{\infty} \{\theta_i(p,\tau)\}_F \cos\left(n \pm \frac{1}{2}\right) \frac{\pi}{R} x .$$

For the transformation of the second derivative we have the formula

$$\left\{\frac{\partial^2 \theta_i(p,\tau)}{\partial x^2}\right\}_F = -\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{R^2}\{\theta_i(p,\tau)\}_F + \\ + (-1)^n \left(n+\frac{1}{2}\right)\frac{\pi}{R}\theta_i(R,\tau) - \frac{\partial \theta_i(0,\tau)}{\partial x}. \qquad (2.11)$$

For the variable τ we apply the Laplace transformation

$$\{\theta_i(x,s)\}_L = \int_0^\infty \theta_i(x,\tau)\, e^{-s\tau}\, d\tau. \qquad (2.12)$$

then

$$\left\{\frac{\partial \theta_i(x,s)}{\partial \tau}\right\}_L = s\{\theta_i(x,s)\}_L - \theta_i(x,0). \qquad (2.13)$$

By the successive application of the integral transformations (2.10), (2.12), *taking into account formulae* (2.11), (2.13), the initial conditions (2.4), and the boundary conditions (2.8), (2.9), the initial problem reduces to an algebraical system, from which, by means of the corresponding inverse integral transformations, we have:

$$\theta_i(x,\tau) = \sum_{j=1} (A^i_{js} v_{js} + B^i_{js} w_{js}) \quad (i,s = 1,2,3). \quad (2.14)$$

Here A^i_{js}, B^i_{js} are certain constants depending on the parameters which enter into the system of equations (2.1)–(2.3), and analogous to the corresponding coefficients given in § 7, Chapter VII:

$$v_{js} = \frac{2}{R}\sum_{n=1}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{R^2}\nu_j \tau\right] \times \\ \times \int_0^R f_s(\xi)\cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}\xi\, d\xi \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x, \quad (2.15)$$

$$w_{js} = \frac{2}{R}\sum_{n=0}^{\infty}{}' \exp\left[\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{R^2}\nu_j \tau\right](-1)^n \times \\ \times \left(n+\frac{1}{2}\right)\frac{\pi}{R} g_s(\tau)\cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x, \qquad (2.16)$$

where

$\nu_j c^2$ are the roots of the cubic equation

$$Ap^3 + Bp^2 + Cp + D = 0\,,$$

$$A = 1 - \varepsilon\,,$$

$$B = c^2\left[(1-\varepsilon)(a_1 + a_3) + \frac{a_2\,\delta\varepsilon\varrho\,c_1}{c_2} + \frac{a_2'\,\varepsilon\gamma_0\,c_2}{c_3} + a_2\right],$$

$$C = c^4\left[a_1 a_3(1-\varepsilon) + a_2 a_3 + a_1 a_2 + \frac{a_2\,\delta\varepsilon\varrho\,c_2\,a_3}{c_1} + \right.$$

$$\left. + a_1 a_2\,\varepsilon\gamma_0\,\frac{c_2}{c_3}\right],$$

$$D = c^6\,a_1 a_2 a_3\,,$$

$$c = \left(n + \frac{1}{2}\right)\frac{\pi}{R}\,.$$

Here it is assumed that the coefficients of the system of differential equations satisfy the following condition:

$$\begin{vmatrix} p + a_1 c^2 & -\frac{\varepsilon\varrho\,c_2}{c_1}\,p & 0 \\ a_2\,\delta\,c^2 & (1-\varepsilon)\,p + a_2 & a_2'\,c^2 \\ 0 & \frac{\varepsilon\gamma_0\,c_2}{c_3}\,p & p + a_3 c^2 \end{vmatrix} \neq 0\,.$$

The symbol $\overset{\tau}{*}$ denotes the convolution operation:

$$f(\tau)\overset{\tau}{*} g(\tau) = \int_0^\tau f(\xi)\,g(\tau-\xi)\,d\xi =$$

$$= \int_0^\tau f(\tau-\xi)\,g(\xi)\,d\xi = g(\tau)\overset{\tau}{*} f(\tau)\,.$$

In order to simplify the subsequent calculations we introduce the theta-function, so that the expressions (2.15) and (2.16) are written thus:

$$v_{js} = \frac{1}{2R}\int_0^R \left\{\vartheta_2\left[\frac{x-\xi}{2R}, \frac{\nu_j\tau}{R^2}\right] + \vartheta_2\left[\frac{x+\xi}{2R}, \frac{\nu_j\tau}{R^2}\right]\right\} f_s(\xi)\,d\xi\,,$$

$$w_{js} = \frac{1}{R}\,\frac{\partial}{\partial x}\int_0^\tau \vartheta_2\left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2}\right] \varphi_s(\eta)\,d\eta\,, \qquad (2.17)$$

$$\vartheta_2(x, \tau) = \frac{1}{\sqrt{(\pi\tau)}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{(x+n)^2}{\tau}\right]. \quad (2.18)$$

We now determine the unknown functions $\varphi_k(\tau)$ $(k = 1, 2)$ from the boundary conditions (2.6) and (2.7). Differentiating (2.14) with respect to x, and replacing the second derivative with respect to x of the theta-function ϑ_2 by its derivative with respect to τ, we obtain

$$\frac{\partial \vartheta_i(x, \tau)}{\partial x} = \sum_{j=1}^{3} (A_{js}^i P_{js} + B_{js}^i Q_{js}) \, (i, s = 1, 2, 3), \quad (2.19)$$

where

$$P_{js}(x, \tau) = \frac{1}{2R} \int_0^R \frac{\partial}{\partial x} \left\{ \vartheta_2 \left[\frac{x-\xi}{2R}, \frac{\nu_j \tau}{R^2}\right] + \right.$$

$$\left. + \vartheta_2 \left[\frac{x+\xi}{2R}, \frac{\nu_j \tau}{R^2}\right] \right\} f_s(\xi) \, d\xi,$$

$$Q_{js}(x, \tau) = \frac{1}{\nu_j R} \int_0^\tau \frac{\partial}{\partial(\tau - \eta)} \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2}\right] \varphi_s(\eta) \, d\eta. \quad (2.20)$$

Taking into account (2.18), it is not difficult to see that the derivative $\partial\vartheta_2/\partial(\tau - \eta)$, which exists for all values of $x \neq R$, tends to infinity, like $\tau^{-3/2}$ as $\tau \to 0$, when $x = R$ and $\tau \to \eta$. In order to avoid this difficulty, we will replace the conditions (2.6) and (2.7) by others which are obtained from them by integration with respect to τ. Instead of (2.6) and (2.7), we then have

$$-a_1 c_1 \gamma_0 \int_0^\tau \frac{\partial \vartheta_1(R, \eta)}{\partial x} \partial\eta + \alpha \int_0^\tau [\theta_{1c} - \theta_1(R, \eta)] \, d\eta -$$

$$- (1 - \varepsilon) \varrho \int_0^\tau q_m(\eta) \, d\eta = 0. \quad (2.21)$$

$$a_2 c_2 \gamma_0 \left[\delta \int_0^\tau \frac{\partial \theta_1(R, \eta)}{\partial x} d\eta + \int_0^\tau \frac{\partial \theta_2(R, \eta)}{\partial x} d\eta\right] +$$

$$+ a_2' c_3 \gamma_0 \int_0^\tau \frac{\partial \theta_3(R. \eta)}{\partial x} d\eta + \int_0^\tau q_m(\eta)\, d\eta = 0\,. \qquad (2.22)$$

Integrating (2.19) with respect to τ, we have

$$\int_0^\tau \frac{\partial \vartheta_i(x, \eta)}{\partial x} d\eta = \sum_{j=1}^{3} (A_{js}^i P_{js}^* + B_{js}^i Q_{js}^*) \quad (i, s = 1, 2, 3)\,, \quad (2.23)$$

where

$$P_{js}^*(x, \tau) = \int_0^\tau P_{is}(x, \eta)\, d\eta, \quad Q_{js}^*(x, \tau) = \int_0^\tau Q_{is}(x, \eta)\, d\eta\,.$$

Integrating (2.20) with respect to τ, and changing the order of integration by means of the Dirichlet formula, we obtain

$$Q_{js}^*(x, T) = \frac{1}{v_j R} \int_0^T d\tau \int_0^\tau \varphi_s(\eta) \frac{\partial}{\partial(\tau - \eta)} \times$$

$$\times\, \vartheta_2 \left[\frac{R - x}{2R}, \frac{v_j(\tau - \eta)}{R^2}\right] d\eta = \frac{1}{v_j R} \times$$

$$\times \int_0^T \varphi_s(\eta)\, d\eta \int_\eta^T \frac{\partial}{\partial(\tau - \eta)} \vartheta_2 \left[\frac{R - x}{2R}, \frac{v_j(\tau - \eta)}{R^2}\right] d\tau =$$

$$= \frac{1}{v_j R} \int_0^T \varphi_s(\eta)\, d\eta \left\{\vartheta_2 \left[\frac{R - x}{2R}, \frac{v_j(\tau - \eta)}{R^2}\right] - \vartheta_2 \left[\frac{R - x}{2R}, 0\right]\right\}.$$

From this expression it follows, that for all values of $x < R$

$$Q_{js}^*(x, T) = \frac{1}{v_j R} \int_0^T \varphi_s(\eta) \vartheta_2 \left[\frac{R - x}{2R}, \frac{v_j(T - \eta)}{R^2}\right] d\eta\,. \quad (2.24)$$

By direct calculation we can show that formula (2.24) holds also for $x = R$, i.e.

$$Q_{js}^*(R, T) = \frac{1}{v_j R} \int_0^T \varphi_s(\eta)\, \vartheta_2 \left[0, \frac{v_j(T - \eta)}{R^2}\right] d\eta\,. \qquad (2.25)$$

The last integral converges, since, according to (2.18), the function ϑ_2 for $\eta \to T$ tends to infinity like $T^{-1/2}$ as $T \to 0$. Forming by means of (2.23)–(2.25) the corresponding terms of expressions (2.21) and (2.22), remembering (2.9), we obtain a system of integral Volterra equations of the first kind:

$$\sum_{\beta=1}^{2} \int_0^{\tau} K_{\alpha\beta}(\tau, \eta)\, q_\beta(\eta)\, d\eta = g_\alpha(\tau) \quad (\alpha = 1, 2), \quad (2.26)$$

where the functions $K_{\alpha\beta}(\tau, \eta)$ and $g_\alpha(\tau)$ have a form which enables us to reduce the system of equations (2.26) to a system of generalized Abel equations, and the latter to a system of integral equations of Volterra of the second kind, as shown in § 7 of Chapter IV:

$$q_\alpha(\tau) + \sum_{\beta=1}^{2} \chi_{\alpha\beta}(\tau, \eta)\, q_\beta(\eta)\, d\eta = \omega_\alpha(\tau) \quad (\alpha = 1, 2). \quad (2.27)$$

The solution of the problem is obtained after finding from (2.27) $q_1(\tau)$ and $q_2(\tau)$ and substituting these in equalities (2.14) and (2.17).

§ 3. The Influence of the Criteria of Similarity on High Intensity Heat and Mass Exchange

The solutions set out above (see §§ 1, 2) show that the distribution of the non-dimensional potentials of the transfer of energy and mass, and also of their rates of change, depend on a large group of criteria of similarity. In addition to the criteria mentioned in Chapter II § 6, high-intensity heat and mass transfer begin to be influenced by new criteria of similarity, characterizing the molar transfer (Bu, Lu_p). The change in the mechanism of transfer, accompanying the onset of molar flux of mass and heat, begins also to show a new effect in the influence of the group of criteria already considered. Thus, the criteria of internal heat and mass exchange (Pn), (Ko) are no longer independent in relation to the fields of certain potentials of transfer.

The criteria of surface heat and mass exchange Bi_q and Bi_m preserve the character of their influence on the heat and mass transfer which has been observed in the case of molecular transfer. Small values of the Biot criteria correspond to undeveloped potential fields, and the transfer of mass and energy takes place slowly. At values of the Biot criteria of the order of a few tenths, there appear in the body large gradients of temperature and mass content, as the result of which the redistribution of mass and heat is intensified. At the same time great stresses appear in the material, producing conditions for warping and for the formation of cracks and pores. As is also the case with molecular transfer, after the establishment of the quasi-regular state, i.e. beginning from $Fo \approx 0{\cdot}1$ to $1{\cdot}0$, the heat exchange criterion of Biot begins to influence only the thermal characteristic (Fig. 53a), whilst the mass exchange criterion of Biot influences only the field of non-dimensional mass content (Fig. 53b). The field of filtrational potential becomes independent of both criteria. The indifference of the Biot criteria to the filtrational potential, and, consequently, also to molar transfer, compels us to suppose that there exists a connection between the criteria of surface heat and mass exchange and the molecular mechanism of transfer. It is clear from Fig. 53 that the field of non-dimensional mass content depends on the criterion B_m less closely than the field of non-dimensional temperature does on the criterion B_q. This is due to the greater inertia of the potential of transfer of mass in comparison with the transfer of heat when $Lu < 1$. The analysis of the solutions obtained also shows that, in a period of steady state, the rates of change of transfer potential of mass and filtrational potential decrease linearly with increasing Θ and do not depend on the value of Bi_q.

In the three-dimensional coordinate space $(\bar{T}, \bar{\Theta}, \bar{P})$ the process of high intensity heat and mass transfer is characterized by a non-self-intersecting curve, each point of which corresponds to its generalized time (Fo). By changing

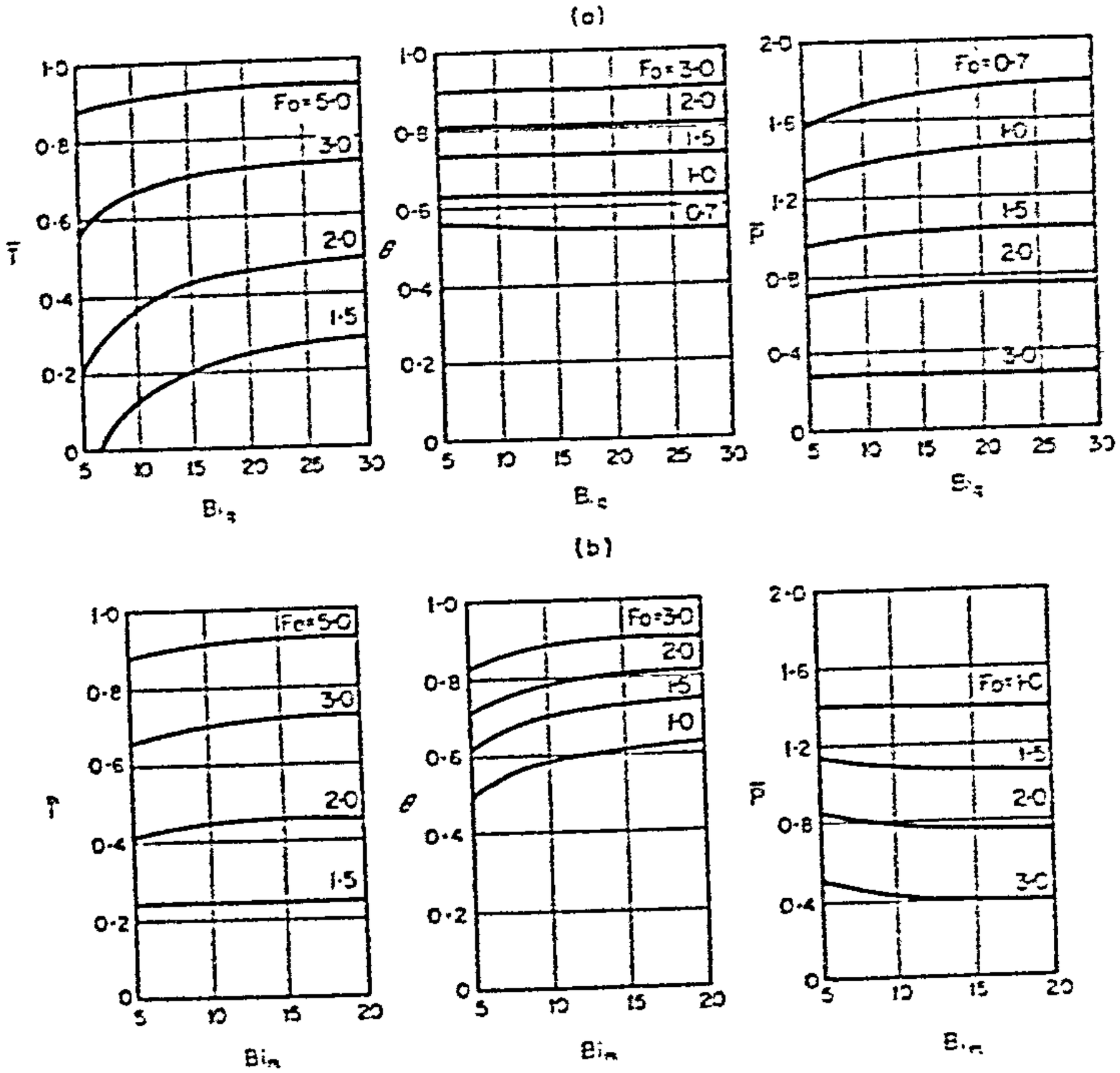


FIG. 53. Relation of the transfer potentials: (*a*) — to Bi_q; (*b*) — to Bi_m

one of the criteria of similarity, we obtain a displacement io the curve of the process, forming the surface of the criterion nf question. The lines obtained by the intersections of the surface of the process with the coordinate surfaces give us the connection between the potentials themselves and the given criterion when the other potentials are constant. The uniqueness of the relations obtained allows us to conclude that the potentials of transfer depend on the criteria of similarity, and on this basis to compare theory with experiment. Thus, for the criteria of surface heat and mass exchange, the character of the change of the criteria with increasing Θ agrees well with the results of the experimental researches of Lebedev and Chizhskii.

The existence of molar transfer in the material, along with molecular transfer of mass, changes the connections between the distribution of non-dimensional potentials and the criteria of inner heat and mass exchange (Pn, Ko). In Chapter II § 6 we did not consider the molar transfer of matter, but the independence of Pn with respect to the temperature field was shown. Molar transfer causes the appearance of new connections between the distribution of temperature and the criterion Pn. For small values of Fo, the temperature of the dispersed medium noticeably increases with increasing Pn, but in course of time this effect diminishes and, for values of the Fourier criterion greater than 2, acquires an opposite character (Fig. 54a). The connection between the filtrational potential and Pn, along with the new character of the action of the criterion on the distribution of temperature, indicates that the Posnov criterion is connected both with the molecular and also with the molar mechanism of transfer. An analogous conclusion may be drawn also with regard to the Kossovich criterion (Fig. 54b). The appearance in the material of the powerful molar transfer is shown in the first place by the redistribution of matter, but a redistribution of heat arises as a consequence of this. The Posnov criterion, which in molecular transfer characterizes the internal mass exchange processes, must reflect the appearance of the new mechanism more sharply than the Kossovich criterion, which characterizes the internal temperature exchange processes. The change which occurs in the character of the influence of the criterion Pn on the heat exchange index of the process turns out to be greater than the change produced in the criterion Ko in relation to the mass exchange index. With increase in the criterion Pn, the non-dimensional transfer potential of mass diminishes. This shows that in the same interval of time, with lower values of the Posnov criterion, a greater quantity of matter is removed from the material. In other words, the greater Pn the more slowly the material dries. The influence of the criterion Ko is first seen in the temperature field, where

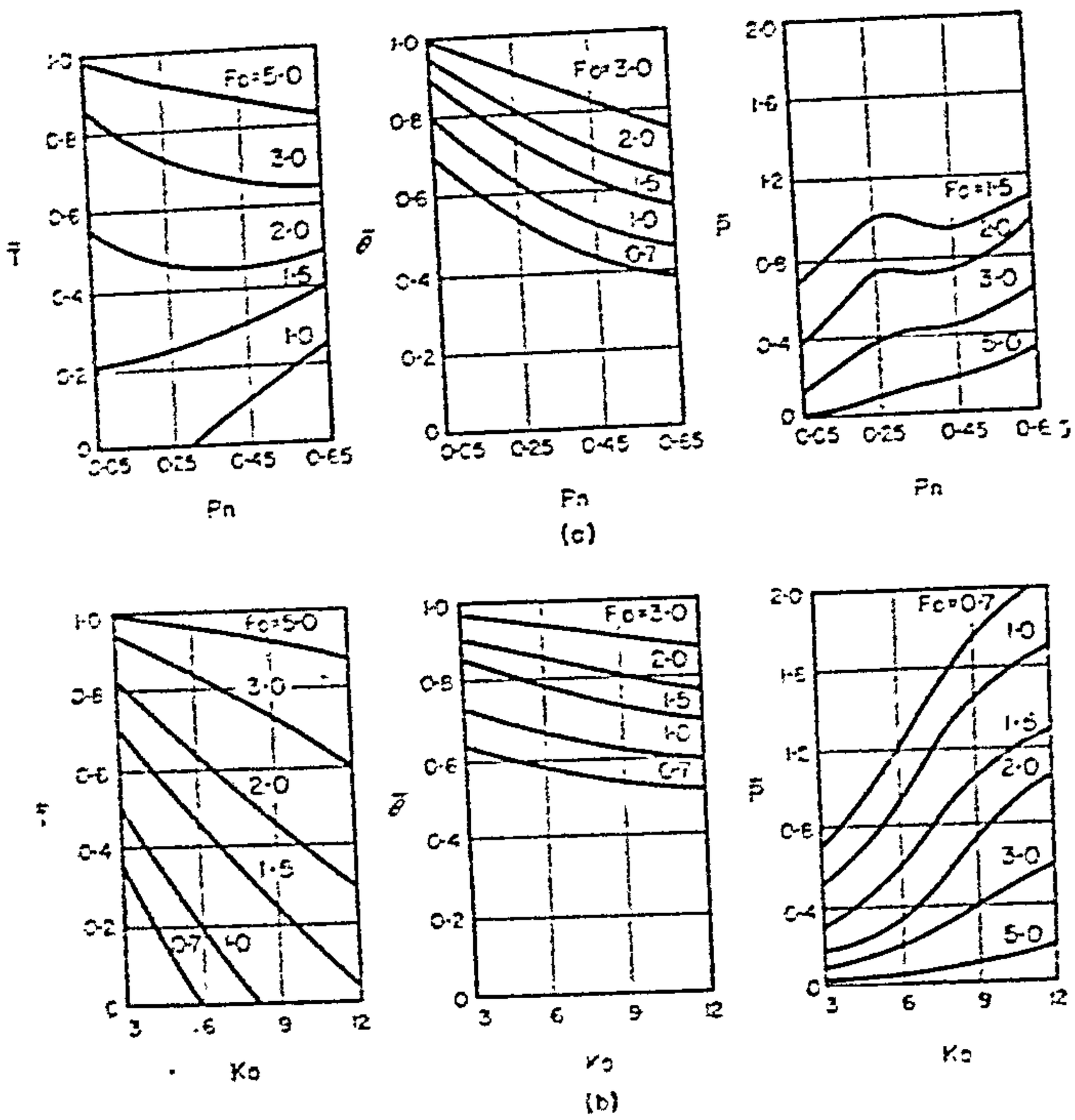


Fig. 54. Relation of the transfer potentials: (*a*) — to Pn; (*b*) — to Ko

the above-noticed similarity of the Kossovich criterion to the heat exchange criteria of Biot is preserved. The reduction of the mean temperature of the material with increase in the criterion Ko, and also the weak influence of this criterion on the intensity of drying, shows that it is necessary to arrange, for bodies subject to thermal decomposition, such a treatment that Ko may have a very large value.

The Bulygin criterion $Bu = (\varrho c_p / c_q) \cdot p_c / (t_c - t_0$ is) new and specific for high intensity heat and mass transfer. It belongs to the group of criteria concerned with internal heat and mass exchange. The Bulygin criterion, like the Kossovich criterion, is characteristic of the accumulation capacity of the body. Unlike the Kossovich criterion it does not represent all the heat expended in evaporation, but only

the heat expended in the production of the vapour, which takes part in molar transfer. The Bulygin criterion is entirely concerned with molar transfer, and hence it influences exclusively the distribution of filtrational potentials through

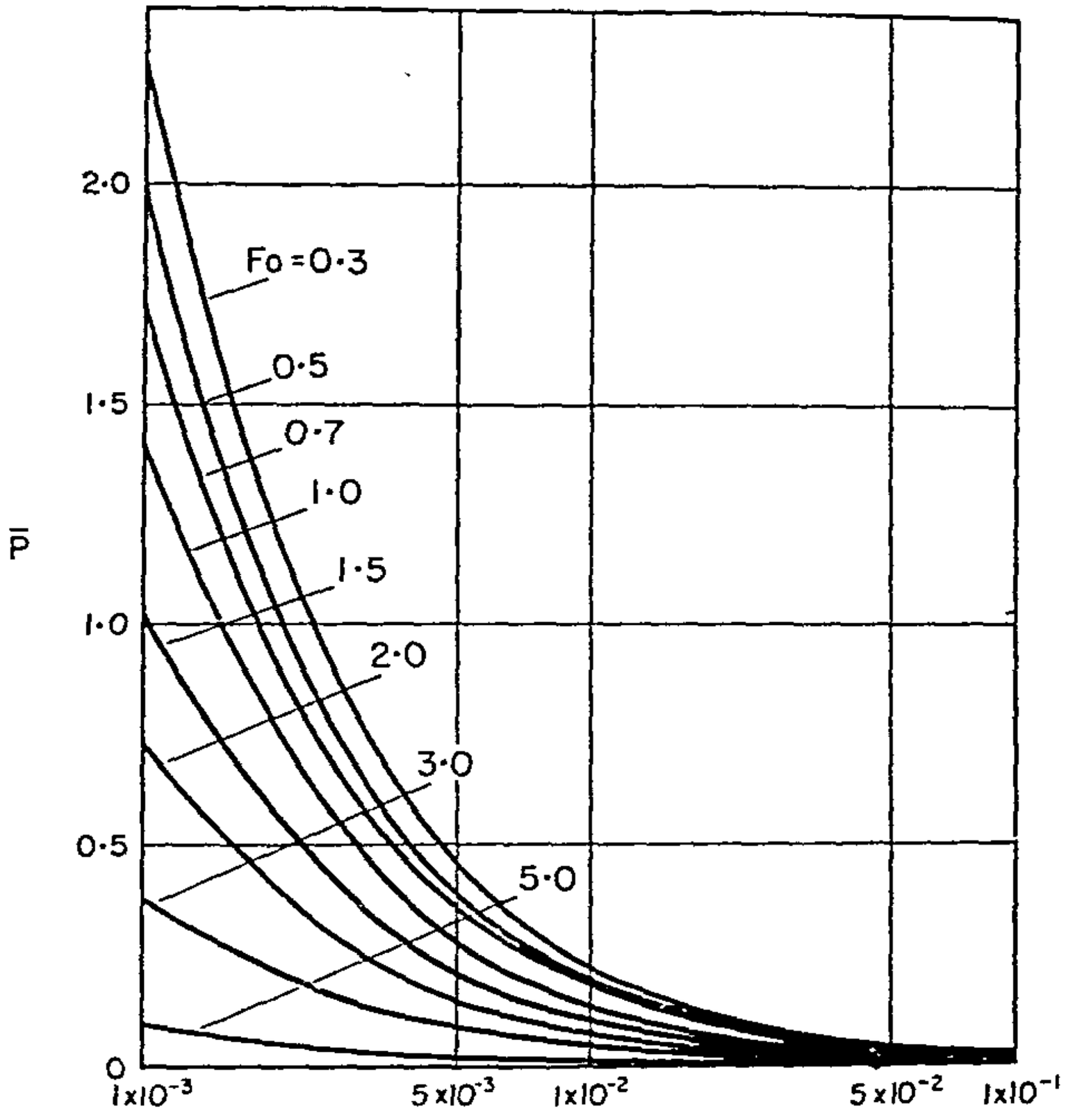


FIG. 55. Relation between the filtrational potential and Bu

the material, and has nothing to do with the distribution of temperature or mass content. The connection between the filtrational potential and the criterion Bu is shown in Fig. 55. The criterion Bu varies through a very wide interval. The magnitude of the mean value of the filtrational potential is inversely proportional to the criterion and diminishes with increase in the latter.

Another criterion of similarity of molar-type for high-intensity heat and mass exchange is Lu_p, which is characteristic of the connection between the molar transfer of mass

proceeding in the material and the transfer of heat, i.e. the inertial quality of the field of filtrational potential in relation to the temperature fileds. In this respect it is analogous to the criterion Lu. It will be seen from Fig. 56 that the criterion Lu_p is independent of T and θ. It varies over a very wide

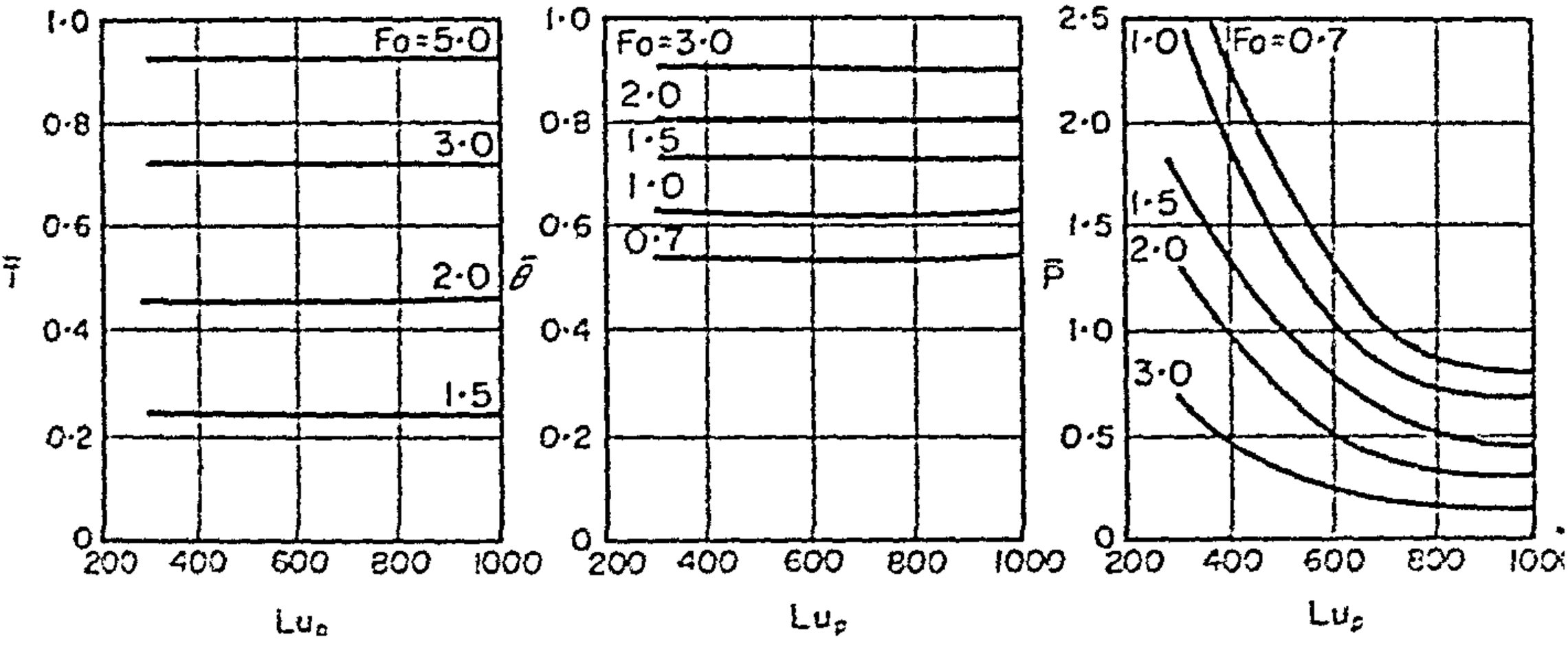


FIG. 56. Relation between the transfer potentials and Lu_p

interval, from some hundreds for convective heating to several tens of thousands for high frequency heating of the moist material. In connection with the analysis of the dependence of heat and mass transfer on the molar criteria of similarity we will consider in somewhat more detail the dynamics of the change of the filtrational potential in general, and in particular, of the nonrelaxed pressure excess.

In the intensive heating of moist dispersed media a great formation of vapour takes place inside the material, leading to the production of a steady gradient of total pressure. This is due to the fact that during the relaxation of the excess pressure through the skeleton of the material, the vapour necessary for the maintenance of the initial condition is produced simultaneously. Depending on the method of heating the medium, and also on the value of the thermophysical and mechanical characteristics of the material, the distribution of filtrational potential can acquire one or another character. Nevertheless in the initial state of the process,

in every case, an intensive increase of the filtrational potential is observed. With reduction of the mass content of the material and increase in θ the intensity of the internal vapour formation is diminished, and this leads to a diminution in the gradient of the filtrational potential. The magnitude of the filtrational potential and the character of its change, as remarked above, is different for different criteria of similarity. To a first approximation, the specific influence of all of these criteria may be estimated by the magnitude of their virtual changes, taking the initial value of each of the criteria of the chosen combination as 100. From Fig. 57 and from calculations not reflected in the graph, it follows that the magnitude of the filtrational potential depends first of all on the magnitude of the criteria Lu, Lu_p and Bu. A decisive role is played by the value of the criterion Lu. Thus, for example, if for $Lu = 0{\cdot}3$, $\bar{P} = 0{\cdot}7268$, then respectively for $Lu = 0{\cdot}7$, we obtain $\bar{P} = 60{\cdot}57$, and for $Lu = 1{\cdot}0$ we have $\bar{P} = 83{\cdot}95$. The characteristic property of the filtrational potential is this, that it is more sensitive to a change in the values of the criteria than transfer potentials of heat and mass. This is the reason for the main difficulty in regulating $\bar{P}$.

On the basis of the analysis carried out above, and also from consideration of Fig. 57, we may propose simplified *criterial equations to describe the high-intensity* heat and mass transfer. The equations are as follows:

$$\begin{aligned} \bar{T} &= \bar{T}(\mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}) , \\ \bar{\Theta} &= \bar{\Theta}(\mathrm{Lu}, \mathrm{Bi}_m, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}) , \qquad (3.1) \\ \bar{P} &= \bar{P}(\mathrm{Lu}, \mathrm{Lu}_p, \mathrm{Bu}, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}) . \end{aligned}$$

Hence it follows that the experimental data on the kinetics of the high intensity heating of capillary-porous bodies must be treated in the form of the criterial relations (3.1). Depending on the parameters of state of the process, certain criteria of heat and mass transfer acquire fundamental (dominating) importance, so that the remaining criteria can be neglected.

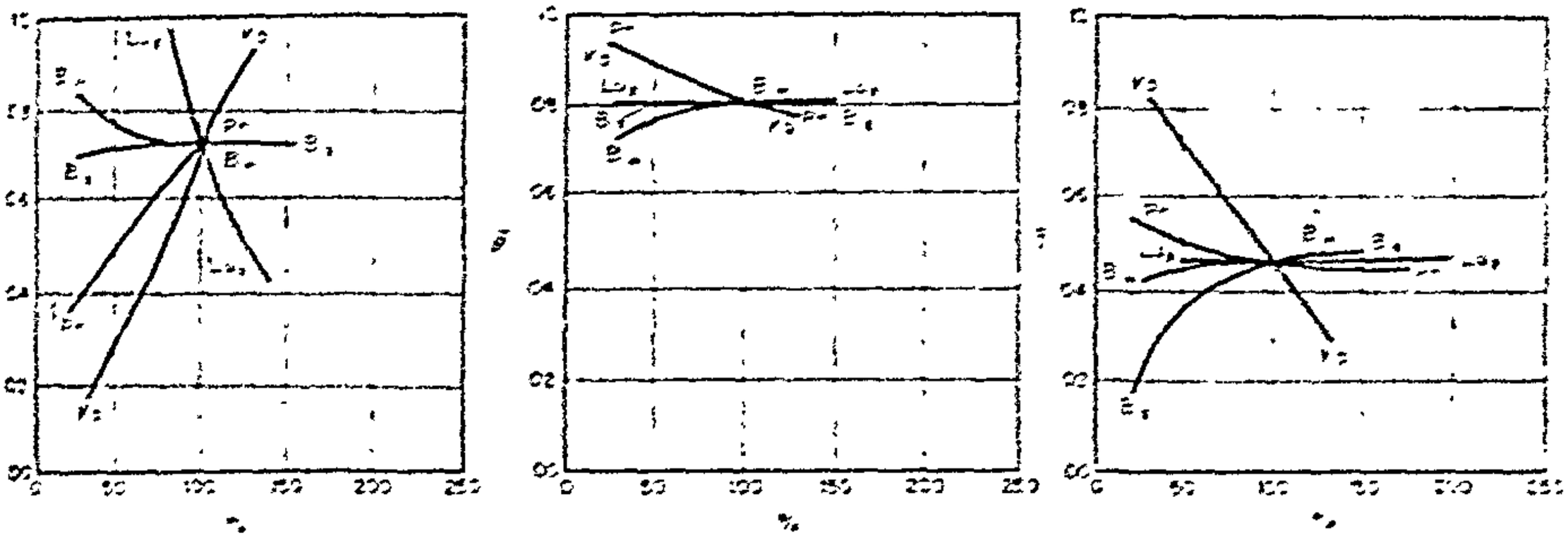


FIG. 57. Specific influence of the criteria of similarity on high-intensity by heat and mass transfer

§ 4. Heat and Mass Transfer on Reduction of Pressure

In a number of branches of industry. in order to intensify the drying process and to harden the material, the drying of moist material with reduction of pressure is beginning to be employed. The rate of drying by this method is several te ns of times greater than that in high temperature convective drying. The transfer phenomena which occur on the reduction of pressure can be satisfactorily explained on the basis of the theory which has been developed.

The essential of the method of drying with reduction of pressure is the maximum use of molar transfer for the dehydration of the material. For this purpose the material is first heated under pressure, and then the pressure is quickly reduced. On account of the accumulated heat in the whole mass of the body, a vigorous evolution of vapour occurs. The molar current of vapour thus formed carries out of the material, along with the vapour, up to 40 per cent of the associated matter in the liquid phase. Consequently, this method of drying proves to be analogous to the wringing out of the material by pressing or centrifuging.

The somewhat changed form of the differential equations (XVIII) for this problem can be written thus:

$$c_q \gamma' \frac{\partial t}{\partial \tau} = \nabla \cdot (\lambda_q \nabla t) + \varepsilon' \varrho c'_m \gamma' \frac{\partial \vartheta}{\partial \tau} - \sum_i c_i q_{mi} \nabla t ,$$

$$c_m \gamma \frac{\partial \theta}{\partial \tau} = \nabla \cdot \left(\sum_{i=1}^{2} \lambda_{pi} \nabla p \right),$$

$$c_{p2} \gamma \frac{\partial p}{\partial \tau} = \nabla \cdot (\lambda_{p2} \nabla p) - m b_T \frac{\partial t}{\partial \tau} - \varepsilon' c'_m \gamma \frac{\partial \theta}{\partial \tau},$$

or in the criterial form for a body in the form of an infinite plate

$$\frac{\partial T(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 T(X, \mathrm{Fo})}{\partial X^2} + \varepsilon' \mathrm{Ko} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}},$$

$$\frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Ra}\, \mathrm{Lu}_p (1 + \lambda_p^*) \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2},$$

$$\frac{\partial P(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Lu}_p \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2} - \mathrm{Le} \frac{\partial T(X, \mathrm{Fo})}{\partial \mathrm{Fo}} -$$

$$- \frac{\varepsilon'}{\mathrm{Ra}} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}}. \qquad (4.1)$$

From physical considerations in developing the system of equations (4.1), we have neglected the molecular transfer of mass. The evaluation of the components of the equations of heat conductivity shows that the transfer of heat by convective current makes up not more than 2·5 per cent of the component due to phase-transformation. Thus, for an accuracy sufficient for practical purposes, we may neglect this also, especially as the solution of the problem is thus considerably simplified.

In the system of equations (4.1) new non-dimensional complexes are introduced and some of those previously used are rendered more precise. Here:

$\mathrm{Ra} = \frac{\gamma c_{p2} p_0}{\gamma c_m \theta_0}$ is the criterion of Ramsin, which characterizes the vapour content of the molar transfer, expressed in fractions of the mass content of matter taking part in the molecular transfer; the Ramsin criterion replaces the Bulygin criterion which we

sidered earlier, and is connected with the latter by the relation Ra = Bu/Ko;

Le = $\frac{mb_T(t_a^0 - t_0)}{\gamma c_2 p_0}$ is the Lebedev criterion; it caracterizes the relation between the flux of molar expansion and the general flux of molar transfer of vapour (m is the porosity of the material), $b_T = (\partial\omega_2/\partial T)_p$ is the intensity of the expansion of the vapour in the capillaries of the body as a result of heating);

ε' = $\varepsilon(1 - \varkappa)$ is the phase-change criterion on reduction of pressure, where $\varkappa$ is the criterion of the thermo-mechanical removal of bound matter in the liquid phase (for $\varkappa = 0$ there is no removal of matter in the liquid phase, for $\varkappa = 1$ the dehydration of the material is completely determined by its thermo-mechanical decompression);

λ_d^* = $\lambda_{p1}/\lambda_{p2}$ is a simplex equal to the ratio of the molar conductivity of matter in the liquid (subscript 1) and the vapour (subscript 2) phases;

Lu_p = a_{p2}/a_q is the criterion of the inertial property ($a_{p2} = \lambda_{p2}/\gamma c_{p2}$ is the coefficient of filtrational potential conductivity of the vapour);

$$T = t_0 - t/t_c^0 - t_0\,,\ \Theta = \theta_0 - \theta/\theta_0\,,\ p = p_0 - p/p_0\,.$$

Depending on the object in view, the method of reduction of pressure can be varied. For example if, in addition to drying, it is desired to harden the material, a lengthy period of reduction of pressure is desirable. If, on the other hand, the principal object is the maximum use of the effect of a reduction of pressure for dehydration then the reduction is carried out quickly, its duration being measured in seconds or in tens of seconds. The character of the pressure reduction and of the associated fall in temperature can be represented approximately by a linear or by an exponential function, i. e.

$$P_c(\tau) = p_0 - e_p\tau \text{ and } t_c(\tau) = t_c^0 - e_T\tau, \text{ or}$$

$$P_c(\tau) = p_0 \exp(-e_p\tau) \text{ and } t_c(\tau) = t_c^0 \exp(-e_T\tau)\,.$$

(a) *Infinite Plate. Duration of the Reduction of Pressure is a Linear Function of the Time.*

The boundary conditions in this problem can be written in the following form:

$$\frac{\partial T(1,\mathrm{Fo})}{\partial X} + \mathrm{Bi}_q[1 + T(1,\mathrm{Fo}) - \mathrm{Pd}_q \cdot \mathrm{Fo}] -$$

$$- (1 - \varkappa - \varepsilon')\,\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*) \times \frac{\partial P(1,\mathrm{Fo})}{\partial X} = 0\,, \quad (4.2)$$

$$P(1,\mathrm{Fo}) = \mathrm{Lu}_p\,\mathrm{Pd}_p\,\mathrm{Fo} \quad (4.3)$$

where $\mathrm{Pd}_p = \frac{\varepsilon_p R^2}{a_{p^2}\,\mathrm{Po}}$ and $\mathrm{Pd}_q = \frac{\varepsilon_T R^2}{a_q(t_0^0 - t_0)}$ are respectively the mass exchange and heat exchange Predvoditelev criteria, characterizing the intensity of the change of pressure and temperature of the medium.

We will assume that the problem is symmetrical, and that the initial conditions are constant, i.e. we will postulate the conditions (1.4) and (1.5).

Applying the integral Laplace transformation to the equation (4.1), and taking account of initial conditions (1.5), we obtain

$$sT_L = T_L'' + \varepsilon'\,\mathrm{Ko}\,s\,\Theta_L\,, \quad (4.4)$$

$$S\,\Theta_L = \mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*)\,P_L'' \quad (4.5)$$

$$SP_L = \mathrm{Lu}_p P_L'' - \mathrm{Le}\,s\,T_L - \frac{\varepsilon'}{\mathrm{Ra}}\,s\,\Theta_L\,. \quad (4.6)$$

The solution of the system of equations (4.4)–(4.6) taking account of condition (1.4) gives us

$$T_L = -\frac{1}{\mathrm{Le}} \sum_{i=1}^{2} B_i(1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)\,\nu_i^2]\cosh \sqrt{(s)}\,\nu_i X, \quad (4.7)$$

$$\Theta_L = \mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*) \sum_{i=1}^{2} B_i\,\nu_i^2 \cosh \sqrt{(s)}\,\nu_i X\,, \quad (4.8)$$

$$P_L = \sum_{i=1}^{2} B_i \cosh \sqrt{(s)}\,\nu_i X\,, \quad (4.9)$$

where

$$\nu_i^2 = \frac{-N + (-1)^{i}\sqrt{(N^2 - 4M)}}{2M} \quad (i = 1, 2) \qquad (4.10)$$

$$M = \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)], \quad N = \mathrm{Lu}_p \times$$

$$\times \left[\varepsilon'(1 + \lambda_p^*)(1 + \mathrm{Ko\,Ra\,Le}) - \frac{1}{\mathrm{Lu}_p} - 1\right].$$

The constants B_i are found from the boundary conditions (4.2)–(4.3). After the application of the Laplace transformation they can be written thus:

$$T'_L(1, S) + \mathrm{Bi}_q\left[\frac{1}{S} - \frac{\mathrm{Pd}_q}{S^2} + T_L(1, S)\right] -$$

$$- (1 - \varkappa - \varepsilon')\,\mathrm{Ko\,Ra\,Lu}_p(1 + \lambda_p^*)\,P'_L(1, S) = 0 \qquad (4.11)$$

$$P_L(1, S) = \frac{\mathrm{Lu}_p\,\mathrm{Pd}_p}{S^2}. \qquad (4.12)$$

Adjusting (4.7)–(4.9) to the conditions (4.11)–(4.12), after a few transformations we obtain:

$$B_1 = -\left[\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p\,Q_2 + \mathrm{Pd}_q \cosh\sqrt{(S)}\,\nu_2}{S^2[Q_1 \cosh\sqrt{(S)}\,\nu_2 - Q_2 \cosh\sqrt{(S)}\,\nu_1]} - \right.$$

$$\left. - \frac{\cosh\sqrt{(S)}\,\nu_2}{S[Q_1 \cosh\sqrt{(S)}\,\nu_2 - Q_2 \cosh\sqrt{(S)}\,\nu_1]}\right],$$

$$B_2 = \frac{\mathrm{Lu}_p\,\mathrm{Pd}_q\,Q_1 + \mathrm{Pd}_q \cosh\sqrt{(S)}\,\nu_1}{S^2[Q_1 \cosh\sqrt{(S)}\,\nu_2 - Q_2 \cosh\sqrt{(S)}\,\nu_1]} -$$

$$- \frac{\cosh\sqrt{(S)}\,\nu_1}{S[Q_1 \cosh\sqrt{(S)}\,\nu_2 - Q_2 \cosh\sqrt{(S)}\,\nu_1]},$$

where

$$Q_i = \frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\,\nu_i^2}{\mathrm{Le}}\left[\cosh\sqrt{(S)}\,\nu_i + \right.$$

$$\left. + \frac{1}{\mathrm{Bi}_q}\sqrt{(S)}\,\nu_i \sinh\sqrt{(S)}\,\nu_i\right] + (1 - \varkappa - \varepsilon')\,\frac{\mathrm{Ko\,Ra\,Lu}_p(1 + \lambda_p^*)}{\mathrm{Bi}_q} \times$$

$$\times \sqrt{(S)}\,\nu_i \sinh\sqrt{(S)}\,\nu_i. \qquad (4.13)$$

Applying to the solution in transform the generalized expansion theorem, we obtain the inverse of the problem in the following form

$$T(X,\mathrm{Fo}) = -\left\{1 - \mathrm{Pd}_q\,\mathrm{Fo} + \frac{1}{2}\,\mathrm{Pd}_q\left(1 - X^2 + \frac{2}{\mathrm{Bi}_q}\right) - \right.$$

$$- (\mathrm{Lu}_p\,\mathrm{Pd}_p + \mathrm{Le}\,\mathrm{Pd}_q)\,\frac{\varepsilon'\,\mathrm{Ko}\,\mathrm{Ra}\,(1+\lambda_p^*)}{2[1-\varepsilon'(1+\lambda_p^*)]}\times$$

$$\left.\times\left(1 - X^2 + \frac{1-\varkappa}{\varepsilon'}\cdot\frac{2}{\mathrm{Bi}_q}\right)\right\} +$$

$$+ \frac{1}{\mathrm{Le}}\sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}(1 - \mathrm{Lu}_p\,[1-\varepsilon'(1+\lambda_p^*)]\,\nu_i^2)\times$$

$$\times \cos\nu_i\mu_n X \exp(-\mu_n^2\,\mathrm{Fo}), \qquad (4.14)$$

$$\Theta(X,\mathrm{Fo}) = \frac{\mathrm{Ra}\,\mathrm{Le}\,(1+\lambda_p^*)}{1-\varepsilon'(1+\lambda_p^*)}\left[-1 - \frac{1}{2}\,\mathrm{Pd}_q\left(1 - X^2 + \frac{2}{\mathrm{Bi}_q}\right) + \right.$$

$$+ \frac{\mathrm{Lu}_p\,\mathrm{Pd}_p + \mathrm{Le}\,\mathrm{Pd}_q}{\mathrm{Le}}\left\{\mathrm{Fo} - \frac{1}{2\,\mathrm{Lu}_p\,[1-\varepsilon'(1+\lambda_p^*)]}\times\right.$$

$$\times\left[1 - X^2 - \varepsilon'\,\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Le}\,\mathrm{Lu}_p(1+\lambda_p^*)\left(1 - X^2 + \right.\right.$$

$$\left.\left.\left.\left. + \frac{1-\varkappa}{\varepsilon'}\cdot\frac{2}{\mathrm{Bi}_q}\right)\right]\right\}\right] - \mathrm{Ra}\,\mathrm{Lu}_p(1+\lambda_p^*)\times$$

$$\times\sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}\,\nu_i^2\cos\nu_i\mu_n X\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (4.15)$$

$$P(X,\mathrm{Fo}) = \mathrm{Lu}_p\,\mathrm{Pd}_p\,\mathrm{Fo} - \frac{\mathrm{Lu}_p\,\mathrm{Pd}_p + \mathrm{Le}\,\mathrm{Pd}_q}{2\,\mathrm{Lu}_p\,[1-\varepsilon'(1+\lambda_p^*)]}(1 - X^2) -$$

$$- \sum_{n=1}^{\infty}\sum_{i=1}^{2} C_{ni}\cos\nu_i\mu_n X\cdot\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (4.16)$$

where

$$C_{n1} = 2\,\frac{\left(1 + \frac{\mathrm{Pd}_q}{\mu_n^2}\right)\cos\nu_2\mu_n + \frac{\mathrm{Lu}_p\,\mathrm{Pd}_p}{\mu_n^2}\,Q_{n2}}{\mu_n\,\Psi_n},$$

$$C_{n2} = -\,2\,\frac{\left(1 + \frac{\mathrm{Pd}_q}{\mu_n^2}\right)\cos\nu_1\mu_n + \frac{\mathrm{Lu}_p\,\mathrm{Pd}_p}{\mu_n^2}\,Q_{n1}}{\mu_n\,\Psi_n},$$

$$\psi_n = \nu_1 A_{n1} \cos \nu_2 \mu_n + \nu_2 Q_{n1} \sin \nu_2 \mu_n - \nu_2 A_{n2} \cos \nu_1 \mu_n - \\ - \nu_1 Q_{n2} \sin \nu_1 \mu_n , \qquad (4.17)$$

$$Q_{ni} = \frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\,\nu_i^2}{\mathrm{Le}} \left(\cos \nu_i \mu_n - \frac{1}{\mathrm{Bi}_q} \nu_i \mu_n \sin \nu_i \mu_n\right) - \\ - (1 - \varkappa - \varepsilon') \frac{\mathrm{Ko\,Ra\,Lu}_p(1 + \lambda_p^*)}{\mathrm{Bi}_q} \nu_i \mu_n \sin \nu_i \mu_n , \qquad (4.18)$$

$$A_{ni} = \frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\,\nu_i^2}{\mathrm{Le}} \sin \nu_i \mu_n + \\ + \left[\frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\,\nu_i^2}{\mathrm{Le}} + (1 - \varkappa - \varepsilon')\,\mathrm{Ko\,Ra\,Lu}_p \times \right. \\ \left. \times (1 + \lambda_p^*)\right] \frac{\sin \nu_i \mu_n + \nu_i \mu_n \cos \nu_i \mu_n}{\mathrm{Bi}_q} . \qquad (4.19)$$

The characteristic roots μ_n are determined from the solution of the equation

$$Q_1 \cos \nu_2 \mu - Q_2 \cos \nu_1 \mu = 0 . \qquad (4.20)$$

(b) *Infinite Plate. Duration of Release of Pressure — an Exponential Function of the Time.*

The boundary conditions for release of pressure according to an exponential law are written thus:

$$\frac{\partial T(1, \mathrm{Fo})}{\partial X} + \mathrm{Bi}_q[1 + T(1, \mathrm{Fo}) + \vartheta(1 - \exp(-\mathrm{Pd}_q' \mathrm{Fo})] - \\ - (1 - \varkappa - \varepsilon')\,\mathrm{Ko\,Ra\,Lu}_p(1 + \lambda_p^*) \frac{\partial P(1, \mathrm{Fo})}{\partial X} = 0 \quad (4.21)$$

$$P(1, \mathrm{Fo}) = 1 - \exp(-\mathrm{Lu}_p \mathrm{Pd}_p' \mathrm{Fo}) , \qquad (4.22)$$

where $\mathrm{Pd}_q' = \frac{c_T R^2}{a_q}$, $\mathrm{Pd}_p' = \frac{c_p R^2}{a_p}$ are respectively the heat exchange and mass exchange Predvoditelev criteria for the exponential case,

$\vartheta = \frac{t_c^0}{t_c^v - t_0}$ is the temperature simplex.

After applying the Laplace transformation to the boundary conditions (4.21) and (4.22) we obtain:

$$T'_L(1,S)+\mathrm{Bi}_q\left[\frac{1+\vartheta}{S}+T_L(1,S)-\frac{\vartheta}{S+\mathrm{Pd}'_q}\right]-$$

$$-(1-\varkappa-\varepsilon')\,\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1+\lambda^*_p)\,P'_L(1,S)=0 \quad (4.23)$$

$$P_L(1,S)=\frac{1}{S}-\frac{1}{S+\mathrm{Lu}_p\,\mathrm{Pd}'_p}. \quad (4.24)$$

The solution of the system of equations (4.4)–(4.6) under boundary conditions (4.23), (4.24) (1.4) and (1.5) gives us:

$$T_L=-\frac{1}{\mathrm{Le}}\sum_{i=1}^{2}\mathrm{Bi}\,(1-\mathrm{Lu}_p\,[1-\varepsilon'(1+\lambda^*_p)]\,\nu_i^2)\cosh\sqrt{(S)}\,\nu_i X,$$

$$\Theta_L=\mathrm{Ra}\,\mathrm{Lu}_p(1+\lambda^*_p)\sum_{i=1}^{2}\mathrm{Bi}\,\nu_i^2\cosh\sqrt{(S)}\,\nu_i X,$$

$$P_L=\sum_{i=1}^{2}\mathrm{Bi}\cosh\sqrt{(S)}\,\nu_i X,$$

where

$$B_1=-\left[\frac{Q_2-(1+\vartheta)\cosh\sqrt{(S)}\,\nu_2}{S\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]}-\right.$$

$$-\frac{Q_2}{(S+\mathrm{Lu}_p\,\mathrm{Pd}'_p)\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]}+$$

$$\left.+\frac{\vartheta\cosh\sqrt{(S)}\,\nu_2}{(S+\mathrm{Pd}'_q)\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]}\right],$$

$$B_2=\frac{Q_1-(1+\vartheta)\cosh\sqrt{(S)}\,\nu_1}{S\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]}-$$

$$-\frac{Q_1}{(S+\mathrm{Lu}_p\,\mathrm{Pd}'_p)\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]}+$$

$$+\frac{\vartheta\cosh\sqrt{(S)}\,\nu_1}{(S+\mathrm{Pd}'_q)\,[Q_1\cosh\sqrt{(S)}\,\nu_2-Q_2\cosh\sqrt{(S)}\,\nu_1]},$$

ν_i and Q_i are determined respectively by the relations (4.10) and (4.13).

Passing to the inversion, we obtain:

$$T(X, \mathrm{Fo}) = -(1+\vartheta) - \frac{1}{Le}\sum_{i=1}^{2}(1-\mathrm{Lu}_p[1-\varepsilon'(1+\lambda_p^*)]\nu_i^2)\times$$

$$\times\{A_i^p\cos[\nu_i\sqrt{(\mathrm{Lu}_p\,\mathrm{Pd}_p)}\,X]\exp(-\mathrm{Lu}_p\,\mathrm{Pd}'_p\,\mathrm{Fo}) -$$

$$-\vartheta\,A_i^q\cos[\nu_i\sqrt{(\mathrm{Pd}'_q)}\,X]\exp(-\mathrm{Pd}'_q\,\mathrm{Fo}) +$$

$$+\frac{1}{\mathrm{Le}}\sum_{n=1}^{\infty}\sum_{j=1}^{2}C_{ni}(1-\mathrm{Lu}_p[1-\varepsilon'(1+\lambda_p^*)]\nu_i^2)\times$$

$$\times\cos\nu_i\mu_n X\exp(-\mu_n^2\,\mathrm{Fo})\}, \qquad (4.25)$$

$$\Theta(X, \mathrm{Fo}) = \frac{\mathrm{Ra}(1+\lambda_p^*)[1-\mathrm{Le}(1+\vartheta)]}{1-\varepsilon'(1+\lambda_p^*)} + \mathrm{Lu}_p\,\mathrm{Ra}(1+\lambda_p^*)\times$$

$$\times\sum_{i=1}^{2}\nu_i^2\{A_i^p\cos[\nu_i\sqrt{(\mathrm{Lu}_p\,\mathrm{Pd}'_q)}\,X]\times$$

$$\times\exp(-\mathrm{Lu}_p\,\mathrm{Pd}'_p\,\mathrm{Fo}) - \vartheta\,A_i^q\cos[\nu_i\sqrt{(\mathrm{Pd}'_q)}\,X]\times$$

$$\times\exp(-\mathrm{Pd}'_q\,\mathrm{Fo})\} - \mathrm{Lu}_p\,\mathrm{Ra}(1+\lambda_p^*)\times$$

$$\times\sum_{h=1}^{\infty}\sum_{i=1}^{2}C_{ni}\nu_i^2\cos\nu_i\mu_n X\cdot\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (4.26)$$

$$P(X, \mathrm{Fo}) = 1 + \sum_{i=1}^{2}\{A_i^p\cos[\nu_i\sqrt{(\mathrm{Lu}_p\,\mathrm{Pd}'_p)}\,X]\times$$

$$\times\exp(-\mathrm{Lu}_p\,\mathrm{Pd}'_p\,\mathrm{Fo}) - \vartheta\,A_i^q\cos[\nu_i\sqrt{(\mathrm{Pd}'_q)}\,X]\times$$

$$\times\exp(-\mathrm{Pd}'_q\,\mathrm{Fo})\} -$$

$$-\sum_{h=1}^{\infty}\sum_{i=1}^{2}C_{ni}\cos\nu_i\mu_n X\cdot\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (4.27)$$

In the solutions (4.25) — (4.27)

$$C_{n1} = 2\,\frac{\dfrac{\mathrm{Lu}_p\,\mathrm{Pd}'_p}{\mu_n^2-\mathrm{Lu}_p\,\mathrm{Pd}'_p}\,Q_{n2} + \dfrac{\mu_n^2-(1+\vartheta)\,\mathrm{Pd}'_q}{\mu_n^2-\mathrm{Pd}'_q}\cos\nu_2\mu}{\mu_n\nu_n},$$

$$C_{n2} = -2\,\frac{\dfrac{\mathrm{Lu}_p\,\mathrm{Pd}'_p}{\mu_n^2-\mathrm{Lu}_p\,\mathrm{Pd}'_p}\,Q_{n1} + \dfrac{\mu_n^2-(1+\vartheta)\,\mathrm{Pd}'_q}{\mu_n^2-\mathrm{Pd}'_q}\cos\nu_1\mu}{\mu_n\nu_n},$$

$$A_1^p = \frac{Q_2^q}{Q_1^q \cos[r_2]'(\mathrm{Lu}_p \mathrm{Pd}_p')] - Q_2^q \cos[r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p')]},$$

$$A_2^p = -\frac{Q_1^q}{Q_1^q \cos[r_2]'(\mathrm{Lu}_p \mathrm{Pd}_p')] - Q_2^q \cos[r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p')]}$$

$$A_1^q = \frac{\cos[r_2]'(\mathrm{Pd}_q')]}{Q_1^q \cos[r_2]'(\mathrm{Pd}_q')] - Q_2^q \cos[r_1]'(\mathrm{Pd}_q')]},$$

$$A_2^q = -\frac{\cos[r_1]'(\mathrm{Pd}_q')]}{Q_1^q \cos[r_2]'(\mathrm{Pd}_q')] - Q_2^q \cos[r_1]'(\mathrm{Pd}_q')]},$$

$$Q_1^p = \frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\, r_1^2}{\mathrm{Lo}} \left\{ \cos[r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p')] - \right.$$

$$\left. - \frac{1}{\mathrm{Bi}_q} r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p') \sin[r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p')] \right\} -$$

$$- (1 - \varkappa - \varepsilon') \frac{\mathrm{Ko}\,\mathrm{Rn}\,\mathrm{Lu}_p(1 + \lambda_p^*)}{\mathrm{Bi}_q} \times$$

$$\times r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p') \sin[r_1]'(\mathrm{Lu}_p \mathrm{Pd}_p')],$$

$$Q_1^q = \frac{1 - \mathrm{Lu}_p[1 - \varepsilon'(1 + \lambda_p^*)]\, r_1^2}{\mathrm{Lo}} \left\{ \cos[r_1]'(\mathrm{Pd}_q')] - \right.$$

$$\left. - \frac{1}{\mathrm{Bi}_q} r_1]'(\mathrm{Pd}_q') \sin(r_1 \mathrm{Pd}_q') \right\} -$$

$$- (1 - \varkappa - \varepsilon') \frac{\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*)}{\mathrm{Bi}_q} r_1]'(\mathrm{Pd}_q') \sin(r_1]' \mathrm{Pd}_q'),$$

r_n, Q_{nl} and A_{nl} are determined respectively by the expressions (4.17), (4.18), (4.19), and the characteristic roots μ_n by the solutions of the equation (4.20).

(c) *Simplified Model of Heat and Mass Transfer with Release of Pressure.*

The expressions obtained for the non-stationary distribution of non-dimensional transfer potentials of mass and heat in the material refer to the general case of drying with release of pressure. The solutions are very cumbersome and, to carry out calculations with them, would involve a great deal of numerical work. In orde rto obtain sufficiently exact analytical relationships, which will be convenient for engineering

practice, it is desirable to simplify further the systems of differential equations (4.1) and the corresponding boundary conditions (4.2), (4.3) and (4.21), (4.22).

First of all we observe that the molecular component of heat transfer in the equation of heat conduction, $a_q \nabla^2 t_1$ are small in comparison with the phase-change term $\varepsilon' \frac{\varrho c_m}{c_q} \frac{\partial \vartheta}{\partial \tau}$. According to experimental data obtained by the present authors it amounts to not more than one per cent of the heat expended in phase-transformation, and may therefore be neglected in calculations sufficiently exact for practical purposes. Moreover, in the release of pressure, the evolution of vapour proceeds substantially only within the material. Hence the criterion of phase transformation $\varepsilon = 1$, and consequently in the boundary conditions, $1 - \varkappa - \varepsilon' = 0$. If at the same time we recognise that on the release of pressure the convective supply of heat removes from the material not more than 0·5 per cent to 0·9 per cent of the associated matter to be dried out, we obtain sufficient grounds for replacing the heat exchange boundary condition of the third kind by a condition of the first kind. However, in the solutions it is sufficient to make use of the boundary conditions (4.3) or (4.22), since the changes of temperature and pressure in the surrounding medium in this case are interdependent. For example, in the case of linear change of pressure (Pd_p) and temperature (Pd_q) with time,

$$\mathrm{Pd}_q = \frac{\varepsilon' \mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*)}{1 - \varepsilon'(1 + \lambda_p^*)\,(1 + \mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Le})} \mathrm{Pd}_p + \frac{1}{\mathrm{Fo}}.$$

Taking into account what has been said, the solution of the simplified system of differential equations under the corresponding boundary conditions gives for the distribution of the non-dimensional potentials of transfer T, Θ and P the following relations:

For a linear type of pressure change:

$$T(X, \mathrm{Fo}) = \frac{\mathrm{Pd}_p(1 - \varkappa)\,\mathrm{Ko}\,\mathrm{Ra}\,(1 + \lambda_p^*)\,\mathrm{Lu}_p^*}{K^2}\left[K\,\mathrm{Fo} - \frac{1}{2}(1 - X^2) + \right.$$

$$+\sum_{n=1}^{\infty}\frac{1}{\mu_n^3}(-1)^{n+1}\cos\mu_n X\cdot\exp(-\mu_n^2 K\,\mathrm{Fo})\Big],$$

$$\Theta(X,\mathrm{Fo})=\frac{\mathrm{Pd}_p\,\mathrm{Ra}\,(1+\lambda_p^*)\,\mathrm{Lu}_p^2}{K^2}\Big[K\,\mathrm{Fo}-\frac{1}{2}(1-X_2)+$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_n^3}(-1)^{n+1}\cos\mu_n X\cdot\exp(-\mu_n^2 K\,\mathrm{Fo})\Big],$$

$$P(X,\mathrm{Fo})=\frac{\mathrm{Pd}_q\,\mathrm{Lu}_p}{K}\Big[K\,\mathrm{Fo}-\frac{1}{2}(1-X^2)+$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_p^3}(-1)^{n+1}\cos\mu_n X\cdot\exp(-\mu_n^2 K\,\mathrm{Fo})\Big].$$

For an exponential pressure change:

$$T(X,\mathrm{Fo})=\frac{(1-\varkappa)\,\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1+\lambda_p^*)}{K}\times$$

$$\times\Bigg[1-\frac{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p}{K}\right)}X}{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{K}\right)}}\exp(-\mathrm{Lu}_p\,\mathrm{Pd}_p'\,\mathrm{Fo})+$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_n}(-1)^{n+1}\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{\mu_n^2 K-\mathrm{Lu}_p\,\mathrm{Pd}_p'}\cos\mu_n X\cdot\exp(-\mu_n^2 K\,\mathrm{Fo})\Bigg],$$

$$\Theta(X,\mathrm{Fo})=\frac{\mathrm{Ra}\,\mathrm{Lu}_p(1+\lambda_p^*)}{K}\times$$

$$\times\Bigg[1-\frac{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{K}\right)}X}{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{K}\right)}}\exp(-\mathrm{Lu}_p\,\mathrm{Pd}_p'\,\mathrm{Fo})+$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_n}(-1)^{n+1}\frac{\mathrm{Lu}_p\,\mathrm{Pd}^p}{\mu_n^2 K-\mathrm{Lu}_p\,\mathrm{Pd}_p'}\cos\mu_n X\cdot\exp(-\mu_n^2 K\,\mathrm{Fo})\Bigg],$$

$$P(X,\mathrm{Fo})=1-\frac{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{K}\right)}X}{\cos\sqrt{\left(\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{K}\right)}}\exp(-\mathrm{Lu}_p\,\mathrm{Pd}_p'\,\mathrm{Fo})+$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_n}(-1)^{n+1}\frac{\mathrm{Lu}_p\,\mathrm{Pd}_p'}{\mu_n^2 K-\mathrm{Lu}_p\,\mathrm{Pd}_p'}\cos\mu_n X.\exp(-\mu_n^2 K\,\mathrm{Fo}).$$

Here

$$K = \mathrm{Lu}_p [1 - (1 - \varkappa)(1 + \lambda_p^*)(1 + \mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Le})],$$

$$\mu_n = (2n - 1)\pi/2, \qquad (n = 1, 2, 3, \ldots).$$

It is clear from the solutions obtained, that there is a definite relation between the non-dimensional potentials T, Θ and P:

$$\Theta(X, \mathrm{Fo}) = \frac{\mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*)}{K} P(X, \mathrm{Fo}),$$

$$T(X, \mathrm{Fo}) = \frac{(1 - \varkappa)\,\mathrm{Ko}\,\mathrm{Ra}\,\mathrm{Lu}_p(1 + \lambda_p^*)}{K} P(X, \mathrm{Fo}).$$

From the two last expressions we can obtain a relation between Θ and T:

$$\Theta(X, \mathrm{Fo}) = \frac{1}{(1 - \varkappa)\,\mathrm{Ko}} T(X, \mathrm{Fo}). \qquad (4.28)$$

If the relation of the criteria τ and Ko to the temperature and mass content (or the transfer potential of mass) is known, then we can determine the local or integral mass content of the material from the value of the non-dimensional temperature. The direct determination and still more the continuous registration of the mass content during the process of pressure release is beset by very great difficulties. However, this can easily be done from formula (4.28). On the other hand, if the distribution of Θ and T in the material is known, then from formula (4.28) we can determine the value of the criterion of phase transformation during the release of pressure, or, more exactly, the value of the criterion of thermo-mechanical removal of bound matter from the material, $\varkappa$.

Conclusion

THE DEVELOPMENT of a new technology and the appearance of new technical processes have shown up the limitations of classical thermodynamics and its inadequacy for the description of the laws of numerous irreversible processes. In many cases these processes have demanded the consideration of the mutually connected and mutually dependent phenomena of transfer, but the solution of such problems by means of the existing macroscopic theories of kinetic processes (for example, the theories of heat conduction, diffusion, electric conduction, etc.) has proved to be impossible, since these were based on experimental laws obtained for individual processes, and in particular, for processes of low intensity.

The foundations of a new theory, which has received the name of the thermodynamics of irreversible processes, have been worked out principally by the researches of Dutch and Belgian scientists. This theory has established connections between different irreversible phenomena, formulated the basic laws of energy and mass transfer, and worked out a method for obtaining the thermodynamic forces and fluxes for different substances. The total effect of all this has been to enable us to proceed to the investigation of the behaviour of macroscopic systems. In essence, this new task involved the review of the kinetic aspect of the question on the basis of the new thermodynamical method, that is the reconsideration of the differential equations of transfer, the working out of methods for solving them, the analysis of the role and influence of the individual factors in the transfer process, with a view to creating conditions for the wider application of the theoretical results in practice and discovering new ways for their further development and generalization.

Among the various transfer phenomena the most widely distributed are those of the transfer of heat and matter. These play a determinative part in most of the processes which occur in the most various power and technological installations, from atomic reactors and rocket motors to the raising of bread in baking.

The present work concludes a series of researches in which the analytical theory of the transfer of heat and matter was worked out for a number of different media. These investigations were carried out principally in the years before 1958, particular attention being paid to disperse media.

The general laws and the systems of differential equations have been formulated for molecular and molar-molecular heat and mass transfer; a method for solving these differential equations by integral transformation has been worked out, and a technique of passing to the limit has been developed which has revealed the connection between the energy and mass characteristics. Making use of the example of heat and mass transfer, we have demonstrated the utility of applying the theory of similitude to the study of transfer phenomena. By several series of numerical calculations we have analysed the influence of the similarity criteria and simplexes on molecular and molar-molecular heat and mass transfer, established the self-modelling nature of individual criteria, established their classification and proposed experimental methods for determining the coefficients and criteria of the similarity of heat and mass exchange from a single experiment. In order to simplify the calculation of kinetically analogous processes, we have introduced many tables and graphs of the characteristic roots and coefficients.

The analytical and experimental researches of the last few years have confirmed the fundamental propositions of the theory. In this connection, we should like to dwell particularly on three questions.

In the first place, the investigations have shown that, in principle, the fundamental propositions of the phenomeno-

logical theory worked out for heat and matter transfer are applicable to a much wider circle of phenomena. For example, transfer phenomena in chemical changes, neutron exchange, the effects connected with thermo-electriciy, etc., can be described by an analogous system of equations.

In the second place, a valid and far-reaching generalization of different connected transfer phenomena can be established, as our researches have shown, on the basis of the theory of similarity. The formulation and solution of the corresponding mathematical models in non-dimensional variables not only generalizes and simplifies the deduction and analysis of the results, but also creates a scientific foundation for the modelling of the non-stationary and mutually connected phenomena of transfer.

In the third place, the use of integral transforms has proved a powerful method for the solution of the systems of differential equations of transfer. Moreover, recent work has foreshadowed still wider prospects for the use of integral transforms in the study of transfer phenomena. Here we will remark only that the basic character of the generalized variables in the image is preserved also in the original. This means that many non-linear equations, relating, for example, to viscous liquids, can be solved so far as is necessary to enable the experimenter to deduce the general relations between the parameters with which he is concerned.

The phenomenological theory of transfer is today passing through a period of rapid development, so that we may expect to witness interesting generalizations and fundamental discoveries during the immediate future.

List of Symbols

$f_L(\xi, s) = \int_0^\infty f(\xi, \tau)\, e^{-s\tau}\, d\tau$ = the Laplace transformation.

t = temperature, deg,

Θ = transfer potential of matter, units of potential,

τ = time, hr,

ξ = generalized coordinate, m,

R = value of the coordinate on the surface of the body,

a_q = coefficient of temperature conductivity, m²/hr,

a_m = coefficient of potential conductivity of mass transfer, m²/hr,

δ = thermo-gradient coefficient, or Soret coefficient, l/deg,

c_q = specific heat capacity of moist body, kcal/kg · deg,

C = heat capacity of heat carrier, kcal/m³ · deg,

γ = density of dry portion of moist body, kg/m³,

v = velocity of heat carrier in layer, m/hr,

P = porosity of layer, or relative pressure,

α_q = coefficient of heat exchange kcal/m² · hr · deg,

α_m = coefficient of mass exchange, kg/m² · hr · potential unit,

β = coefficient of mass exchange, relative to the difference of specific mass contents, m/hr,

p = specific heat of phase change, kcal/kg,

Γ = form factor,

$T = \frac{t(\xi, \tau) - t(\xi, 0)}{t_c - t(\xi, 0)}$ = non-dimensional temperature,

$\Theta = \frac{\theta(\xi, \tau) - \theta(\xi, 0)}{\theta(\xi, 0) - \theta_p}$ = non-dimensional mass transfer potential,

$\mathrm{Fo} = \frac{a_q \tau}{R^2}$ = Fourier criterion,

$\mathrm{Lu} = \frac{a_m}{a_q}$ = criterion of inertia,

$\mathrm{Bi}_q = \frac{\alpha_q}{\lambda_q} R$, $\mathrm{Bi}_m = \frac{\alpha_m}{\lambda_m} R = \frac{\beta R}{a_m}$ = heat exchange and mass exchange criteria of Biot,

$\mathrm{Pn} = \frac{\delta [t_c - t(0)]}{u(0) - u_c} = \frac{\delta_s [t_c - t(0)]}{[\theta(0) - \theta_p]}$ = Posnov criterion,

$\mathrm{Ko} = \frac{\varrho [u(0) - u_p]}{c_q [t_c - t(0)]} = \frac{\varrho c'_m [\theta(0) - \theta_p]}{c_q [t_c - t(0)]}$ = Kossovich criterion,

ε = phase change criterion,

$\mathrm{Fe} = \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} = \dfrac{\varepsilon\varrho\delta}{c_q}$ = Fedorov criterion,

$\mathrm{Mg} = \dfrac{a_q}{vc_p\gamma_p}$ = Margoulis criterion,

$\mathrm{Mn} = \dfrac{FR}{P}$ = Miniovich criterion,

$\mathrm{Pe} = \dfrac{vR}{a_q}$ = Peclet criterion,

$\mathrm{Pd} = \dfrac{\delta R^2}{a_q[t_c - t(\xi, 0)]}$, $\mathrm{Pd}' = \dfrac{\chi R^2}{a_q}$ = Predvoditelev criterion,

$W = \dfrac{t_n - t_u}{t_c - t_n}$; $V = \dfrac{\theta_u - \theta_n}{\theta_r - \theta_p}$ = parametric criteria of non-uniformity of initial distribution of temperature and mass transfer potential,

$W_c = \dfrac{t_c}{t_c - t_0}$; $W_0 = \dfrac{t_0}{t_c - t_0}$ = parametric temperature criteria,

$K_1 = \dfrac{1-\varepsilon}{\varepsilon}\,\mathrm{Lu}\,\dfrac{\mathrm{Bi}_q}{\mathrm{Bi}_m}$, $K_2 = V - \mathrm{Pn}\,\Theta$ = complex criteria,

$\mathrm{Bu} = \dfrac{\varrho\, c_p\, p_0}{c_q(t_c - b_0)}$ = Bulygin criterion.

Subscripts

L = indicates Laplace operational transformation,
F = indicates Fourier operational transformation,
o = initial state,
c = parameter of surrounding medium,
n = parameter of surface,
u = parameter of centre of body,
p = equilibrium value,
q = heat exchange parameter,
m = mass exchange parameter,
j = subscript having two values: $j = 1;\ 2.$

Bibliography

Monographs and Collected Papers

Akopyan, A. A. *General Thermodynamics (Obshchaya Termodinamika).* Moscow-Leningrad, Gosenergoizdat, 1955.

All-union scientific-technical conference for the intensification of processes and improvement of the quality of materials in drying in the principal branches of industry and agriculture (Vsesoyuznoye nauchno-tekhnicheskoye soveshchaniye po intensifikatsiyi protsessov i uluchsheniyu kachestva materiyalov pri sushke v osnovnykh otraslyakh promyshlennosti i sel'skogo khozyaĭstva). Moscow, Profizdat, 1958.

Bosworth, R. C. L., *Processes of Heat Transfer.*

Chechetkin, A. V. *High-temperature Heat Carriers (Vysokotemperaturnye Teplonositeli).* Moscow—Leningrad, Gosenergoizdat, 1957.

Chudnovskiĭ, A. F., *Heat Exchange in Disperse Media (Teploobmen v dispersnykh sredakh).* Moscow, GITTL, 1954.

Crank, I. *Mathematics of Diffusion.* Oxford, 1956.

Denbigh, K. G. *Thermodynamics of Stationary Irreversible Processes.* Methuen, 1951.

Ditkin, V. A. and Prudnikov, A. P. *Operational Calculus with two Variables and its Applications (Operatsionnoye Ischisleniye po Dvum Peremennym i yego Prilozheniya).* Fizmatgiz, 1958. English translation in preparation by Pergamon Press.

D'yakonov, G. K. *Questions of the Theory of Similarity in the Region of Physico-chemical Processes (Voprosy teoriyi podobiya v oblasti fiziko-khimicheskikh protsessov).* Moscow, Izd. Akad. Nauk SSSR, 1956.

Eckert, E. R. and Drake, R. M. *Heat and Mass Transfer.* New York, 1959.

Frank—Kamenetskiĭ, D. A. *Diffusion and Heat Transmission in Chemical Kinetics (Diffuziya i teploperedacha v khimicheskoi kinetike).* Moscow—Leningrad, Izd. Akad. Nauk SSSR, 1947.

Grew, K. E., and Ibbs, G. L. *Thermal Diffusion in Gases.*

Grinberg, G. A. *Select Problems in the Mathematical Theory of Electric and Magnetic Phenomena (Izbrannyye voprosy matematicheskoi teoriyi elektricheskikh i Magnitnykh Yavleniĭ).* Izd. Akad. Nauk SSSR, 1948.

de Groot, S. R. *Thermodynamics of Irreversible Processes.*

GUGGENHEIM, E. A. *Thermodynamics*. Amsterdam, 1957.

GUREVICH, L. *Foundations of Physical Kinetics (Osnovy fizicheskoi kinetiki)*. Moscow—Leningrad, GITTL, 1940.

KATYS, G. P. *Methods and Instruments for Measuring the Parameters of Non-stationary Thermal Processes (Temperature, Pressure, Yield) (Metody i pribory dlya izmereniya nestatsionarnykh teplovykh protsessov—Temperatura, davleniye, raskhod)*. Moscow, Mashgiz, 1959.

KAY, T. M. *Introduction to Fluid Mechanics and Heat Transfer*. Cambridge, 1957.

KICHKINA, YE. S. *Drying and Heat-Exchange Processes in Layers of Material in Lumps (Protsessy sushki i teplo-obmena v sloye kuskovykh materialov)*. Dissertation, LPI, 1947.

KONAKOV, P. K. *Theory of Similarity and its Application in Heat Technology (Teoriya podobiya i yee primeneniye v teplotekhnike)*. Moscow—Leningrad Gosenergoizdat, 1959.

KONDRAT'EV, G. M. *The Regular Thermal State (Regulyarnyĭ teplovoi rezhim)*. Moscow, GITTL, 1954.

KONDRAT'EV, G. M. *Thermal Measurements (Teplovyye izmereniya)*. Moscow—Leningrad, Mashgiz, 1957.

KOSHLYAKOV, N. S. *Fundamental Differential Equations of Mathematical Physics (Osnovnyye differentsial'nyye uravneniya matematicheskoi fiziki)*. ONTI, 1936.

KRYLOV, A. N. *On Certain Differential Equations of Mathematical Physics which have Application to Technical Problems (O nekotorykh differentsial'nykh uravneniyakh matematicheskoi fiziki, imeyushchikh primeneniye v tekhnicheskykh voprosakh)*. Izd. Akad. Nauk SSSR, 1938.

KUTATELADZE, S. S. *Foundations of the Theory of Heat Exchange (Osnovy teoriyi teploobmena)*. Moscow—Leningrad, Mashgiz, 1957.

KUZNETSOV, S. P. *Heating and Cooling of Moist Bodies (Nagrevaniye i okhlazhdeniye vlazhnykh tel)*. Dissertation, 1952.

LANDAU, L. D., and LIFSHITS, E. M. *Mechanics of Continuous Media (Mekhanika sploshnykh sred)*. Moscow, GITTL, 1953.

LEBEDEV, P. D. *Drying with Infrared Rays (Sushka infrakrasnymi luchami)*. Gosenergoizdat, 1955.

LEVICH, V. G. *Physico-Chemical Hydrodynamics (Fiziko-khimicheskaya gidrodinamika)*. Moscow, Fizmatgiz, 1959.

LEVIN, V. I., and GROSBERG, YU. I. *Differential Equations of Mathematical Physics (Differentsial'nye uravneniya matematicheskoĭ fiziki)*. Gostekhizdat, 1951.

LUIKOV, A. V. *Theory of Thermal Conduction (Teoriya teploprovodnosti)*. Gostekhizdat, 1952.

LUIKOV, A. V. *Transfer Phenomena in Capillary-Porous Bodies (Yavleniya perenosa v kapillyarnoporistykh telakh)*. Gostekhizdat, 1954.

LUIKOV, A. V. *Heat and Mass Exchange in the Drying Proces (Teplo- i masso-obmen v protsessakh sushki)*. Gosenergoizdat, 1956.

LUIKOV, A. V. and MIKHAĬLOV, YU. A. *Theory of Energy and Mass Transfer (Teoriya perenosa energii i veshchestva)*. Izd. Akad. Nauk BSSR, Minsk, 1959.

LUIKOV, A. V. (Ed.) *Heat and Mass Exchange in Food Products*. Edited by Professor A. V. Luikov, Doctor of Technical Sciences, Moscow, Pishchepromizdat, 1956. *(Teplo- i massoobmen v pishchevykh produktakh)* (Moskovskiĭ tekhnolog. in-t pishchevoĭ pr-ti, Kafedra fiziki. Trudy, Pt. 6).

LUIKOV, A. V. (Ed.). *Heat and Mass Exchange in Capillary-Porous Bodies*. Collected papers of the Physics Faculty. (*Teplo- i massoobmen v kapillyarno-poristykh telakh*. Sbornik rabot kafedry fiziki). Moscow-Leningrad, Tekhnol. in-t Pishchevoĭ pr-ti, Pt. 8.

LUIKOV, A. V. (Ed.). *Heat and Mass exchange in evaporation processes. (Teplo- i masso-obmen v protsessakh ispareniya, Sbornik statei)*. Moscow, Izd. Akad. Nauk SSSR, 1958.

MIKUSIŃSKI, J. *Operational Calculus* (tr. from Polish) Pergamon Press, 1960.

PRIGOGINE, I. *Introduction to the Thermodynamics of Irreversible Processes (Étude Thermodynamique de Phénomènes Irréversibles)*. Desoer, Liège.

PRIGOGINE, I. *Problems of Energetics (Problemy Energetiki)* Izd. Akad nauk SSSR. Moscow. (Krzhizhanovskii Institute) 1959. *Questions of Mass Transfer* (Scientific-methodological Conference, 1956) *(Voprosy massoperecachi*. Nauchnometodicheskaya konferentsiya, 1956). Leningrad, Goskhimizdat, 1957.

SHEVEL'KOV, V. L. *Thermophysical Characteristics of Insulating materials (Teplofizicheskiye kharakteristiki izolyatsionnykh materialov)*. Moscow—Leningrad, Gosenenergoizdat, 1958.

SHIROKOV, M. F. *Physical Bases of Gas Dynamics and its Application to the Processes of Heat Exchange and Friction (Fizicheskiye osnovy gazodinamiki i primeneniye yee k protsessam teploobmena i treniya)*. Moscow, Fizmatgiz, 1958.

SMIRNOV, M. S. *Application of the Method of Integral Transforms to Problems of the Theory of Molecular Transfer (Primeneniye metoda integral'nykh preobrazovanniĭ k resheniyu zadach teoriyi molekulyarnogo perenosa)*. Dissertation, 1955.

SOKOL'SKIĬ, B. M. *External Heat and Mass Exchange in the Convective*

Drying Process (Vneshnii teplo- i masso-obmen v protsesse konvektivnoi sushki). Minsk. Izd. Belgosuniversiteta, 1957.
SNEDDON, I. *Fourier Transforms*. McGraw-Hill, 1943.
SOMMERFELDT, A. *Thermodynamics and Statistical Physics*.
TIKHONOV, A. N. and SAMARSKIĬ, A. A. *The Equations of Mathematical Physics* English translation by Pergamon Press, 1963.
TOLSTOV, G. P. *Fourier Series (Ryady Fur'e)*. Gostekhizdat, 1951.
TRANTER, C. J. *Integral Transforms in Mathematical Physics*. Methuen, 1951.
ZABRODSKII, S. S. *Transfer of Heat by a Fluidized Bed of Granular Material (Perenos tepla psevdoozhizhennym sloyem zernistovo Materiala)*. Moscow, 1958 (Trudy Instituta energetiki AN BSSR, No. 8).

PAPERS

AĬZENSHTADT, V. S. and METEL'SKII, A. S. Numerical Laplace transformation (Chislennoye preobrazovaniye Laplasa). *IFZh.* 4, No. 2, 1961, pp. 82—91.
BAB'EV, N. N. Simultaneous determination of the coefficients of transfer of heat and moisture in moist materials (Sovmestnoye opredeleniye koeffitsientov perenosa tepla i vlagi vo vlazhnykh materialakh). *Trudy Mosk. Tekhnol. in-ta pishchevoi pr-ti*, 1956, Pt. 6.
BERLYAND, O. S. A method of solving the diffusion (heat conduction) equation [Ob odnom metode resheniya uravneniya diffusii (teploprovodnosti)]. *IFZh.* 2, No. 5, 1959, pp. 70—87.
BERMAN, L. D. An analogy between heat and mass exchange (Ob analogiyi mezhdu teplo- i massobmenom). *Teploenergetika*, 1955, No. 8, pp. 10—19.
CHUDNOVSKII, A. F. Influence of the variable character of the thermophysical characteristics of soils through their profiles on the temperature of the soil (Vliyaniye peremennogo kharaktera teplofizicheskikh kharakteristik pochv po ikh profilyu na temperatury pochvy). *IFZh.* No. 7, 1960, pp. 51—60.
GAMAYUNOV, N. I. A new method of determining the coefficients of heatand mass-exchange (Novyi metod opredeleniya koeffitsientov teplo- i massoobmena). *IFZh.* 2, No. 11, 1959, pp. 35—42.
GAMAYUNOV, N. I. Investigation of the transfer of heat and moisture in a finite rod (Issledovaniye perenosa tepla i vlagi v ogranichennom sterzhne) *IFZh.* 3, No. 4, 1960, pp. 11—17.
GAMAYUNOV, N. I. On the criteria of non-isothermal transfer (O kri-

teriyakh neizotermicheskogo perenosa). *IFZh.* 3, No. 12, 1960, pp. 58–62.

GAVRILOVA, R. I. and PRUDNIKOV, A. P. On a problem in the theory of heat conduction (Ob odnoĭ zadache teoriyi teploprovodnosti). *IFZh.* 3, No. 5, 1960, pp. 136–7.

IVANOVA, L. P. and KIM, E. I. Two-dimensional problem of heat- and mass-exchange in drying processes (Dvukhmernaya zadacha teplo- i masso-obmena v protsessakh sushki). *Izv. Akad. Nauk SSSR, Otd. Tekhn. n. Energetika i Avtomatika,* 1959, No. 3, pp. 76–84.

KOZLOVA, M. S. Analytical investigation of heat and mass exchange in an infinite plate and a sphere (Analiticheskoye issledovaniye teplo- i massoobmena v neogranichennoĭ plastine i share). *Tr. Mosk. Tekhnolog. in-ta Pishchevoĭ Pr-ti,* 1960, Pt. 15, pp. 74–81.

KUKUSHKIN, A. V. New method of determining the coefficients of substance transfer from one experiment (Novyĭ metod opredeleniya koeffitsientov perenosa veshchestva iz odnogvo opyta). *IFZh.* No. 5, 1959, pp. 46–55.

KUKUSHKIN, A. V. New method of determining coefficients of mass transfer (O novom metode opredeleniya koeffitsientov masso-perenosa). *IFZh.* No. 2, 1960, pp. 90–94.

LOGINOV, L. I. and YUSHKOV, P. P. Numerical integration of the system of equations of heat and mass exchange by means of implicit formulae (Chislennoye integrirovaniye sistemy uravneniĭ teplo- i masso-obmena s pomoshch'yu neyavnykh formul). *IFZh.* 3, No. 10, pp. 93–96.

LÝKOV, A. V. A new method of determining the coefficients of temperature conductivity in moist materials (Novyĭ metod opredeleniya koe'ffitsientov temperaturoprovodnosti vlazhnykh materialov). *ZhETF* No. 2, 1935.

LUIKOV, A. V. Fundamental Laws of Heat and Mass Exchange in the Drying Process (Osnovnye zakonomernosti teplo- i masso-obmena v protesse sushki). *Izv. VTI,* No. 10, 1952.

LUIKOV, A. V. Heat and mass exchange in disperse media with phase change (Teplo- i masso-obmen v dispersnykh sredakh pri fazovykh prevrashcheniyakh). *IFZh.* No. 6, 1958.

LUIKOV, A. V., and SHIRKOV, M. S. Investigation of the kinetics of the drying process (K issledovaniyu kinetiki protsessa sushki). *Izv. Akad Nauk SSSR, Otd. Tekhn. n.* No. 8, 1956.

LUIKOV, A. V., and PRUDNIKOV, A. P. Investigation of transfer phenomena in porous media (K issledovaniyu yavlenii perenosa tepla i veshchestva v poristikh sredakh). *Dokl. Akad. Nauk BSSR,* 2, 334–7, 1958.

LUIKOV, A. V. Theory of regular thermal state (K teorii teplovovo regulyarnovo rezhima). *Dokl. Akad. Nauk BSSR*, 1, No. 2, 1957, pp. 52—56.

LUIKOV, A. V. Investigation of transfer phenomena in evaporation processes (Issledovaniye yavlenii perenosa v protsessakh isparcniya). *Tr. Mosk. Tekhnolog. in-ta Pishchevoĭ Pr-ti*, Pt. 8, 1957. pp. 237—45.

LUIKOV, A.V. and POLONSKAYA, F.M. Determination of coefficients of mass transfer in colloidal and capillary-porous bodies with variable moisture content (Opredeleniye koeffitsientov perenosa massy v kolloidnykh i kapillyarno-poristykh telakh s peremennoĭ vlazhnost'yu). *Tr. Vses. Nauch.-issled. Kinofotoinstituta*, Pt. 2, 1958, pp. 113—27.

LUIKOV, A. V. Application of Onsager's theory to investigation of heat and substance transfer (Primeneniye teorii Onzagera k issledovaniyu perenosa tepla i veshchestva). *Tr. in-ta Energetiki Akad Nauk BSSR*, Pt. 4, 1958, pp. 58—67.

LUIKOV, A. V. Fundamental problems of structural thermophysics (Osnovnye problemy stroitel'noĭ teplofiziki). *Izv. Akad. Str. i Arkh., SSSR*, 1959, No. 4, pp. 73—87.

LUIKOV, A. V. Investigation of heat and mass exchange in binary gas mixtures (K issledovaniyu teplonoi masso-obmena v binarnykh gazovykh smesyakh). *IFZh.* 2, No. 11, 1959, pp. 20—28.

LUIKOV, A. V. Molar-molecular mass and heat transfer in capillary-porous bodies (Molyarno-molekulyarnyĭ masso i teploperenos v kapillyarno-poristykh telakh). *IFZh.* 3, No. 10, 1960, pp. 88—92.

MIKHAĬLOV, YU. A. Application of the system of differential equations of heat and mass exchange to the process of convective drying (Primeneniye sistemy differentsial'nykh uravnenii teplo- i masso-obmena k protsessu konvektivnoĭ sushki). *Izv. Akad. Nauk Latv. SSR*, No. 12, 1956.

MIKHAĬLOV, YU. A. Influence of the criteria of similarity on heat and mass exchange in convective drying (Vliyaniye kriteriyev podobiya na teplo i masso-obmen pri konvektivnoi sushke). *Izv. Akad. Nauk Latv-SSR*, No. 6, 1957.

MIKHAĬLOV, YU. A. Analytical investigation of heat and mass exchange in convective drying. (Analiticheskiye issledovaniya teplo- i masso-obmena pri konvektivnoi sushke). In collection: *Teplo- i masso-obmen v protsessakh isparcniya*, 1958, pp. 145—53.

MIKHAĬLOV, YU. A. Theory of convective drying process (K teorii protsessa konvektivnoĭ sushki) *IFZh.* No. 1, 1958.

MIKHAĬLOV, YU. A. Theory of high temperature drying of moist

materials (K teorii vysokotemperaturnoĭ sushki vlazhnykh materialov). *Izv. Akad. Nauk Latv. SSR*, 1958, No. 3, pp. 127—38.

Mikhaĭlov, Yu. A. Heat and mass exchange in drying with superheated steam (Teplo- i masso-obmen pri sushke peregretym parom vody). *Tr. In-ta Energetiki i Elektrotekhniki. Akad. Nauk. Latv. SSR*, No. 8, 1959, pp. 75—100.

Mikhaĭlov, Yu. A. Heat and mass exchange on sudden fall of pressure (Teplo- i masso-obmen pri sbrose davleniya). *IFZh.* 4, No. 2, 1961, pp. 33—43.

Mochalin, A. I. Application of the Dirac function to the solution of problems of heat conduction (Primeneniye funktsii Diraka k resheniyu zadach teploprovodnosti). *IFZh.* 1, No. 5, 1958, pp. 76—83.

Mochalin, A. I. Combined use of the Dirac function and integral transforms (Sovmestnoye ispol'zovaniye funktsii Diraka i integral'nykh preobrazovanii.) *IFZh.* 2, No. 10, 1959, pp. 76—81.

Murashko, M. G. New ideas on the process of freezing moist soils (Novyye predstavleniya o protsesse promerzaniya vlazhnykh gruntov). *IFZh.* 1, 1958, No. 1, pp. 96—99.

Polonskaya, F. M. Investigation of the temperature field of moist materials in the drying process (period of constant rate) (Issledovaniye temperaturnovo polya vlazhnykh materialov v protsesse sushki (period postoyannoĭ skorosti)). *ZhTF.* No. 5, 1953.

Polonskaya, F. M. Application of Onsager's theory to the investigation of the transfer of heat and substance in porous bodies (Primeneniye teorii Onzagera k issledovaniyu perenosa tepla i veshchestva v poristykh telakh) *IFZh.* 1, No. 4, 1958, pp. 87—89.

Polushkin, A. A. On the criteria of similarity of heat and mass exchange in the process of evaporation of a liquid (O kriteriyakh podobiya teplo- i massо-obmena v protsessakh isparеniya zhidkosti) (Review) *IFZh.* 2, No. 2, 1959, pp. 129—44.

Porkhayev, A. P. Application of Onsager's theory to the investigation of the processes of heat and mass exchange (Primeneniye teorii Onzagera k issledovaniyu protsessov teplo- o massoobmena). *Tr. Mosk. tekhnolog. in-ta pishchevoĭ pr-ti* Pt. 8. 1957, pp. 246—9.

Porkhayev, A. P. and Smirnov, M. S. Problem of temperature oscillation in soils (Zadachi o temperaturnykh kolebaniyakh v pochvakh i gruntakh). Collected papers. *Vecs. zochnovo in-ta pishchevoĭ pr-ti*, Pt. 4, 1957.

Prudnikov, A. P. Analytical investigation of the processes of heat

and mass exchange (Analiticheskoye issledovaniye protsessov teplo- i masso-obmena) (Dissertation) *Akad. nauk BSSR*. 1957.

PRUDNIKOV, A. P. Solution of certain problems of the theory of molecular transfer for bodies of very simple geometrical form (Resheniye nekotorykh zadach teorii molekulyarnovo perenosa dlya tel prosteĭshei geometricheskoi formy). *Izv. Akad. nauk SSSR. Otd. tekh. n.*, No. 8, 1957.

PRUDNIKOV, A. P. Solution of a mixed boundary problem of the theory of thermal diffusion (Resheniye odnoĭ smeshannoĭ krayevoĭ zadachi teorii termodiffuzii). *Dokl. Akad. nauk SSSR*. 120, No. 2, 1958, pp. 249—51.

PRUDNIKOV, A. P. Solution in integral form of a mixed problem for systems of two differential equations of parabolic type (Resheniye v integral'noĭ forme odnoĭ smeshannoĭ zadachi dlya sistemy dvukh differentsial'nykh uravnenii parabolicheskovo tipa). *Dokl. Akad. nauk SSSR*. 115, No. 5, 1957, pp. 869—71.

PRUDNIKOV, A. P. Investigation of heat and mass exchange in disperse media (K issledovaniyu teplo- i massoobmena v dispersnykh sredakh). *IFZh*. No. 4, 1958.

RABINOVICH, G. D. Heat and mass exchange in a layer of moist material (Teplo- i masso-obmen v sloye vlazhnovo materiala). *IFZh*. 3, No. 12, 1960, pp. 29—34.

SHIMKO, N. G., and YUSHKOV, P. P. On a finite integral transform of Hankel (Ob odnom konechnom integral'nom preobrazovanii Khankelya). *IFZh*. 2, No. 6, 1959.

SIZYAKOVA, E. I. Investigation of the kinetics of thermal moisture conduction in capillary-porous bodies. (Issledovaniye kinetiki termovlago-provodnosti v kapillyarnoporistykh telakh). *Tr. Mosk. tekhnolog. in-ta pishchevoĭ pr-ti*, Pt. VI, 1956.

SIZYAKOVA, E. I. Analytical investigation of internal heat and mass exchange in capillary-porous bodies with phase change (Analiticheskoye issledovaniye vnutrennevo teplo- i masso-obmena v kapiolyarno-poristykh telakh pri nalichii fazovykh prevrashcheniĭ). *Tr. Mosk. tekhnolog. in-ta pishchevoĭ pr-ti*, Pt. 15, 1960, pp. 117—26.

SMIRNOV, M. S. Non-stationary potential fields of heat and moisture transfer in presence of evaporation and boundary conditions of the second kind. (Nestatsionarnye polya potentsialov perenosa tepla i vlagi pri nalichii ispareniya i granichnykh uslovii II-vo roda). *Tr. Vscs. nauchn.—issled. kinofoto—instituta*, Pt. 2, 1958, pp. 128—38.

SMIRNOV, M. S. Problem of heat conduction with variable thermal characteristics of the material (Zadacha teplo-provodnosti

pri peremennykh termicheskikh kharakteristikakh materiala). *Tr. Mosk. tekhnolog. in-ta pishcheroĭ pr-ti* Pt. 11, 1958, pp. 250—3.

TEMKIN, A. G. Influence of the integral criterion of form on the process of heat conduction (Vliyaniye integral'novo kriteriya formy na protsess teploprovodnosti). *ZhTF*. Pt. 3, 1954.

TEMKIN, A. G. Geometrical criterion of cooling (Geometricheskiĭ kriteriĭ okhlazhdeniya). *ZhTF*, No. 4, 1952.

TEMKIN, A. G. Inertia of temperature fields (Inertsiya temperaturnykh poleĭ). *IFZh*. 3, No. 1, 1960, pp. 68—75.

TEMKIN, A. G. Inertia of temperature fields. (Inertsiya temperaturnykh poleĭ). *IFZh*. 2, No. 12, 1959. pp. 11—19.

TEMKIN, A. G. Temperature field of moveable heat sources (Temperaturnoye pole podvizhnykh istochnikov tepla). *Dokl. Akad. Nauk BSSR*. 4, No. 2, 1960, pp. 55—57.

Tsoĭ, P. V. Problem of the transfer of heat and moisture for a semi-bounded three-dimensional medium under boundary conditions of the second kind. (Zadacha o perenose tepla i vlagi dlya poluogranichennoĭ sredy trekh izmereniĭ pri krayevykh usloviyakh vtorovo roda). *IFZh*. 3, No. 6. 1960, pp. 112—19.

Tsoĭ, P. V. Problem of the transfer of heat and moisture in the presence of evaporation and boundary conditions of the second kind (Zadacha o perenose tepla i vlagi pri nalichii ispareniya i krayevykh usloviĭ vtorovo roda). *IFZh*. 3, No. 12, 1960, pp. 53—57.

YABLONSKAYA, V. P. Investigation of the transfer of heat and moisture in freezing and thawing soils. (K issledovaniyu perenosa tepla i vlagi v promerzayushchikh i protaivayushchikh gruntakh). *I.F.Zh*. 3, 1960, No. 1, pp. 91—93.

YERMAKOV, V. S. Investigation of the processes of heat transfer in the heat-evolving elements of a nuclear reactor (Issledovaniye protsessov perenosa tepla v teplo-vydelyayushchikh elementakh yadernogo reaktora). *IFZh*. 1, No. 9, 1958, pp. 3—15.

YERMOLENKO, V. D. Application of the methods of the thermodynamics of irreversible processes to the phenomena of mass transfer in colloidal bodies (Primeneniye metodov termodinamiki neobratimykh protsessov k yavleniyam massoperenosa v kolloidnykh telakh). *IFZh*, No. 5, 1960, pp. 124—7.

YERMOLENKO, V. D. Investigation of mass transfer in colloidal bodies (K issledovaniyu massoperenosa v kolloidnykh telakh). *IFZh*. No. 8, 1960. pp. 117—20.

YUSHKOV, P. P. Approximate solution of problems of non-stationary heat conditions by the method of finite differences. (Pribliz-

hennoye resheniye zadach nestatsionnarnoi teploprovodnosti metodom konechnykh rasnostei). *Tr. in-ta Energetiki. Akad. Nauk. BSSR.*, Pt. 6, 1958, pp. 3—158.

YUSHKOV, P. P. Numerical integration of the equation of heat conductivity in the case where the thermal coefficients depend on the temperature (O chislennom integrirovanii uravneniya teploprovodnosti v sluchae, kogda termicheskiye koeffitsienty zavisyat ot temperatury). *IFZh.* 1, No. 9, 1958, pp. 102—8.

YUSHKOV, P. P. Numerical integration of the system of differential equations of the transfer of heat and mass (O chislennom integrirovanii sistemy differentsial'nykh uravnenii perenosa tepla i massy vechestva). *Tr. In-ta Energetiki, Akad. Nauk, BSSR*, Pt. 10, 1959, pp. 73—80.